What is the Role of the Arms of a Linear Broadcast Antenna?

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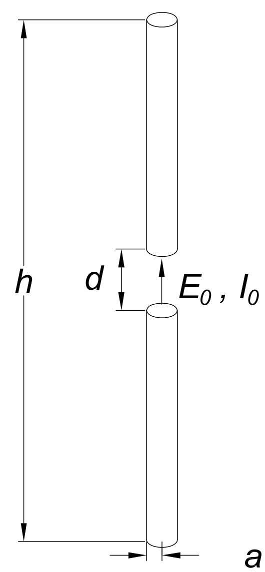
1 Problem

A broadcast antenna is a transducer of energy from a voltage source into the energy of distant electromagnetic fields. In its simplest form the antenna could consist of a compact voltage source located in the gap between two linear conductors (the "arms"), as sketched below.



The voltage source maintains oscillatory voltage $V_0\, e^{i\omega t}$ between its terminals, which are also the ends of the two linear conductors of radius a and total length h. The voltage source can deliver whatever current I_0 is required to maintain electric field E_0 in the gap of length d, where $V_0 = E_0 d$.

The voltage source, including its terminals, is a small electric dipole oscillator with moment $p_0 = Q_0 d = V_0 C_0 d$, where C_0 is the capacitance of the terminals.[1] In the absence of the arms of the antenna, this oscillating dipole emits time-average power P_0 into the far zone, where (in Gaussian units; see, for example, sec. 67 of [2]),

$$P_0 = \frac{|\ddot{p}|^2}{3c^3} = \frac{p_0^2 \omega^4}{3c^3} = \frac{V_0^2 C_0^2 d^2 \omega^4}{3c^3} \approx \frac{V_0^2 a^2 d^2}{12 c^3}\,, \tag{1}$$

c is the speed of light in vacuum, and the approximation holds if $a \ll d$. All media in this problem can be assumed to have unit relative permittivity and permeability.

Discuss the role of the arms of the antenna in enhancing the emitted power. Do these arms "radiate" the emitted power?

[1] As discussed in [1], the capacitance C_0 is approximately $a/2(1 - a/d)$ when $a < d$.

2 Solution

The power emitted by the antenna comes from the voltage source. The form of eq. (1) suggests that the arms of the antenna can increase the power output by increasing the dipole moment and the capacitance of the system.

When the total length h of the arms is small compared to the wavelength $\lambda = 2\pi c/\omega$, the electric dipole moment of the system is $p_0 = -iI_0 h/2\omega$, and the time-average emitted power is (see, for example, p. 191 of [3]),

$$P = \frac{\pi^2 \left| I_0 \right|^2 h^2}{3c\lambda^2} = \frac{\left| I_0 \right|^2 \omega^2 h^2}{12c^3} . \tag{2}$$

The peak current I_0 is related by,

$$I_0 = \frac{V_0}{Z} \approx iV_0\,\omega C \approx \frac{iV_0\,\omega h}{8\ln(h/2a)} , \tag{3}$$

noting that the (terminal) impedance Z of a small linear antenna is almost purely reactive, and that the capacitance is (see, for example, sec. 2.1 of [4]),

$$C \approx \frac{h}{8\ln(h/2a)} . \tag{4}$$

Then, the emitted power is,

$$P = \frac{V_0^2 \omega^4 h^4}{768 c^3 \ln^2(h/2a)} , \tag{5}$$

which is large compared to the power (1) emitted by the voltage source alone whenever $h \gg d$.

Equation (2) indicates an additional role of the arms in increasing the "radiation resistance", $R_{\rm rad}$, of the antenna, defined by,

$$P = \frac{\left| I_0 \right|^2 R_{\rm rad}}{2}, \qquad \textit{such that} \qquad R_{\rm rad} = \frac{2\pi^2 h^2}{3c\lambda^2} = \frac{20\pi^2 h^2}{\lambda^2}\,\Omega \qquad (h \ll \lambda), \tag{6}$$

recalling that $1/c = 30\ \Omega$.

As the length h of the antenna approaches $\lambda/2$ the inductance of the antenna becomes significant, and for $h \approx \lambda/2$ the reactance X vanishes, and $R_{\rm rad} \approx 71\ \Omega$.

2.1 Do the Arms Scatter the Energy from the Voltage Source?

While the above solution is very straightforward, many antenna enthusiasts feel that it does not give sufficient credit to the arms of the antenna as a more direct "cause" of the increase in the emitted power.

Some people desire a description in which the arms of the antenna can be thought of as emitting the power P. However, as noted as least as early as 1904 (p. 459 of [5]), the requirement that the tangential component of the electric field vanish at the surface of a good/perfect conductor implies that the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{7}$$

(which quantifies the flow of energy in the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ [6]) has no component perpendicular to the surface of the conductor. That is, perfect conductors neither emit nor absorb electromagnetic energy (and good conductors absorb energy that is dissipated as Joule heating). This fact has been called the "radiation paradox" by Schelkunoff [7].

We are led to consider the (good/perfect) conductors of the antenna as guiding the flow of energy from the voltage source into the far zone, rather than generating that energy. A useful view of a rectangular waveguide is that the transmission of energy down the guide can be thought of as due to reflections of slanted waves off the walls [8]. This raises the question of whether the flow of energy in an antenna can be regarded as involving reflections off the conductors of the energy that flows from the voltage source (see, for example, [9][2]).

However, the time-average flow of energy from the voltage source, which is a small electric dipole oscillator of moment p, is purely radial (see, for example, [11]),

$$\langle \mathbf{S} \rangle_{\text{voltage source}} = \frac{p^2\,\omega^4 \sin^2\theta}{8\pi c^3 r^2}\,\hat{\mathbf{r}}. \tag{8}$$

Thus, the energy flow from the voltage source alone never intercepts the linear conductors, and cannot be said to reflect off them.

It seems more proper to say that the voltage source "radiates" the power, or that the antenna system (source plus arms) "radiates", but not that the arms "radiate."[3,4]

A Appendix: Other Decompositions, Other Difficulties

In mathematical analysis of intricate problems it is common to split the problem into pieces that can be solved separately. Then, a full solution can sometimes be obtained by combining the partial solutions, particularly if the original problem is linear. The question then arises

[2]The key assumption of [9] is not stated there, but (K. Macleish, private communication) is that the electric field should be decomposed into its so-called "rotational" and "irrotational" parts (see, for example, [10]), which are computed from the instantaneous values of the electric field throughout the entire Universe. However, the partial fields $\mathbf{E}_{\text{rot}}$ and $\mathbf{E}_{\text{irr}}$ of this elegant mathematical formalism do not have separate physical significance. As emphasized by Faraday, there is only one physical electric field, $\mathbf{E}_{\text{total}}$. See also Appendix A.2.

[3]The term "radiation" is not crisply defined in the literature of electromagnetism. The author has come to consider "radiation" to be the flow of energy described by the Poynting vector (7) [12]. This generalizes the IEEE definition 2.309 [13]: "Radiation, electromagnetic. The emission of electromagnetic energy from a finite region in the form of unguided waves," to include guided waves as well.

[4]An insightful statement by R.W.P. King [14] is *With reference to the question as to what is the source of the radiant energy, the answer is simply, the generator. Energy is transferred from the generator to the antenna by the transmission line or wave guide. The antenna is that part of the conducting surfaces on which are the currents and charges that are used to calculate the radiation field.*

whether the partial solutions found in this mathematical process can or should be regarded as having separate physical significance. For many people, the fact that the partial solutions contribute to a full solution of a physical problem gives the partial solutions a kind of physical significance.

In this Appendix, I will be more skeptical, and apply this skepticism to decompositions of the electric and magnetic fields **E** and **B**, and of the related Poynting vector **S** of eq. (7).

A.1 Electrostatic and Electrokinetic Fields

When Faraday first studied effects of time-dependent currents he found that they led to phenomena that seemed related to that associated with electric fields, which had previously been a static concept. A transcription of this into vector notation is that the electric field (and the magnetic field) can be related to a scalar potential V and a vector potential $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{9}$$

Should we suppose that there are actually two physical electric fields, defined by,

$$\mathbf{E}_1 = -\boldsymbol{\nabla} V, \qquad \mathbf{E}_2 = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{10}$$

such that $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$?

Faraday used little mathematical analysis, and instead remarked that when measuring the force on a test charge, and attributing that force to an electric field, one can deduce only a single field in this manner, the total electric field **E**. As such, he argued against supposing that there are multiple electric fields at a given point in space, all with separate physical significance.

With time, it was noted that the potentials V and $\mathbf{A}$ are not unique, in the sense than an infinite set of potentials can all correspond to the same fields **E** and **B**. This lack of uniqueness is now the more common objection to supposing that the partial fields of eq. (10) have separate physical significance.[5]

A recent "rediscovery" of the decomposition (10) appears in [16].

A.2 The Helmholtz Decomposition

An important mathematical decomposition of any vector field **F** (that is suitably well behaved) was noted in 1858 by Helmholtz [17],[6]

$$\mathbf{F} = \mathbf{F}_{\rm irr} + \mathbf{F}_{\rm rot}, \tag{11}$$

where the irrotational and rotational components $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$ obey[7]

$$\boldsymbol{\nabla} \times \mathbf{F}_{\rm irr} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{F}_{\rm rot} = 0. \tag{12}$$

[5] A historical review of so-called gauge transformations of the potentials is given in [15].

[6] The essence of this decomposition was anticipated by Stokes (1849) in secs. 5-6 of [18].

[7] The irrotational component is sometimes labeled "longitudinal" or "parallel", and the rotational component is sometimes labeled "solenoidal" or "transverse".

In particular,

$$\mathbf{F}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}', \qquad \text{and} \qquad \mathbf{F}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{\nabla}' \times \mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}', \tag{13}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Time does not appear in eq. (13), which indicates that the vector field $\mathbf{F}$ at some point $\mathbf{r}$ (and some time t) can be reconstructed from knowledge of its vector derivatives, $\boldsymbol{\nabla} \cdot \mathbf{F}$ and $\boldsymbol{\nabla} \times \mathbf{F}$, over all space (at the same time t). A major historical significance of the Helmholtz decomposition (11) and (13) was in showing that Maxwell's equations, which give prescriptions for the vector derivatives $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$, are mathematically sufficient to determine the field $\mathbf{E}$ (and similarly for the field $\mathbf{B}$).

However, the partial fields $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$ involve instantaneous action at a distance and should not be regarded as physically real. They are only wonderfully convenient mathematical constructs.

As was noted in footnote 2, the decomposition of the electric field of antennas in [9] is based on that of Helmholtz'.

A.3 Decomposition into Canonical, Orbital and Spin Terms

An interesting set of decompositions arises from considerations of charged particles as having canonical momenta and angular momenta due to their interactions with external electromagnetic fields. This permits assignment of some of the momentum and angular momentum of the electromagnetic fields to the canonical momenta and angular momenta of the charges, resulting in suggestive forms of the remaining field momentum and angular momentum [19, 20].

The field decomposition begins with the Helmholtz decomposition, which implies computations based on instantaneous knowledge throughout the Universe, such that the interpretation of the various terms in this decomposition remains problematic. However, the resulting decomposition of angular momentum suggests that one can identify a "spin" angular momentum in classical electromagnetism, which catches the fancy of many readers. Perhaps it will be no surprise that the interpretation of this classical "spin" angular momentum is ambiguous. For the author's commentary on this, see sec. 2.4.2 of [20].

A.4 Cross Terms in the Decomposition of the Poynting Vector

Decompositions of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are mathematically convenient because these fields obey linear differential equations. Any two solutions add to form another solution, and it is easy to suppose that the linear sum retains the separate significance of the two partial solutions.

However, the Poynting vector is a quadratic function of the fields, so that the Poynting vector of a sum of partial fields is not equal to the sum of the Poynting vectors of the partials fields. If we write $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, then,

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi}(\mathbf{E}_1 + \mathbf{E}_2) \times (\mathbf{B}_1 + \mathbf{B}_2) \\ &= \frac{c}{4\pi}\mathbf{E}_1 \times \mathbf{B}_1 + \frac{c}{4\pi}\mathbf{E}_2 \times \mathbf{B}_2 + \frac{c}{4\pi}(\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1) \\ &\equiv \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_{12}, \end{aligned} \tag{14}$$

where,

$$\mathbf{S}_1 = \frac{c}{4\pi}\mathbf{E}_1 \times \mathbf{B}_1, \tag{15}$$

$$\mathbf{S}_1 = \frac{c}{4\pi}\mathbf{E}_2 \times \mathbf{B}_2, \tag{16}$$

$$\mathbf{S}_{12} = \frac{c}{4\pi}(\mathbf{E}_1 \times \mathbf{B}_2 + \mathbf{E}_2 \times \mathbf{B}_1). \tag{17}$$

Hence, any decomposition in which both the electric and magnetic fields are represented in terms of partial fields will lead to a cross term $\mathbf{S}_{12}$ in the Poynting vector, whose interpretation is not clear.

This issue is avoided in decompositions only of the electric field, such as those considered in secs. A.1 and A.2, whose lack of physical significance has other reasons. The decompositions considered in the remainder of this Appendix all involved partitions of both the electric and magnetic fields with various mathematical appeal, but all with lack of physical clarity when considering the Poynting vector (and other quadratic functions of the fields such as densities of energy, momentum and angular momentum).

A.5 Decomposition into "Radiation" and "Nonradiation" Fields

In problems involving emission of power to "infinity" it is common to decompose the electric and magnetic fields into "radiation" and "nonradiation" fields. The "radiation" fields are computed from the derivatives of the retarded current densities, weighted by the inverse of the retarded distance from the source point to the observer.

This type of analysis was first performed in the frequency domain, as represented, for example, in [21, 22], while in the time domain the relevant decomposition is [23],

$$\mathbf{E}(\mathbf{x}, t) = \int \frac{[\rho]\,\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + ([\mathbf{J}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R}\,d\mathbf{x}'. \tag{18}$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{n}}}{R}\,d\mathbf{x}', \tag{19}$$

where $R = |\mathbf{R}|$ with $\mathbf{R} = \mathbf{x} - \mathbf{x}'$, $\hat{\mathbf{n}} = \mathbf{R}/R$, and a pair of brackets, [ ], implies the quantity within is to be evaluated at the retarded time $t' = t - R/c$. The "radiation" fields are the last terms in eq. (18)-(19).

While this decomposition appears appealing, it does not lead to crisp understanding except in the far zone, and this author does not advocate its use in problems where one is interested in characterizing the flow of energy in the near zone. Rather, it is best to consider the total Poynting vector $\mathbf{S}$, with no attempt to decompose it into partial energy flows. For further commentary, see [12].

A.6 Decomposition into Plane Electromagnetic Waves

An appealing decomposition of general, time-dependent electromagnetic fields is into a sum of plane electromagnetic waves of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. This decomposition must include the possibility that the wave vector $\mathbf{k}$ is complex, which corresponds to either (well-known) attenuated waves or (less well-known) evanescent waves close to some relevant surface in the problem.

Technical details of such decompositions have been presented in [24, 25, 26].

Since plane electromagnetic waves extend over all space and all time, this type of decomposition offers little insight into what goes on in the vicinity of, say, the arms of an antenna system.

A.7 Decomposition into "In" and "Out" Fields

A technique familiar from optics is the decomposition of the fields into an "in" field and an "out" field. The success of such a decomposition depends on being able to ignore cross terms between the "in" and "out" fields, which is generally valid for observers far from any matter in the problem.[8]

A.7.1 The "In" Fields Are Those of the Voltage Source of an Antenna

As discussed above in sec. 2.1, in the case of a linear antenna the "in" fields of the voltage source are associated with a purely radial Poynting vector, which implies that no energy of the "in" fields is incident on the conductors of the antenna. The total energy flow remains parallel to the surface of the conductors of the antenna, so the energy flow associated with the "out" fields and with the cross terms is also parallel to the surface of the conductors.

In the far zone, the pattern of the flow of energy is slightly different than that of the voltage source alone, but no insight into this small difference is obtained by considerations of the "in" and "out" fields on taking the "in" fields to be those associated with the voltage source alone.

A.7.2 The "Out" Fields Are Those of a Short Segment of a Linear Antenna

In electrostatics, an understanding of the electrical force on a surface element of a (perfect) conductor can be obtained by decomposing the field above that element into the field due to the surface charge density σ on that element plus the electric field due to all other charges in the problem. Namely, the electric field just above the surface is due in equal parts to the surface charge on the element itself and to the rest of the Universe, each part being $\mathbf{E} = 2\pi\sigma\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

This suggests that we consider a decomposition of the fields of a linear antenna into those associated with a small segment of the antenna (the "out" fields) plus those due to all other charges and currents (the "in" fields).[9] For this, we must also decompose the magnetic field just above the surface element. A less familiar argument based on Maxwell's fourth equation and the surface current density $\mathbf{K}$ tells us that the magnetic field just above the surface is

[8] If the observer is directly in line with the "in" wave, interference between the "in" and "out" fields cannot be ignored, which leads to phenomena such as the Arago dark spot, which astonished members of the Académie Français around 1818.

[9] This decomposition was suggested in sec. VI of [7], where no claim was made as to its merits towards understanding energy flow.

due in equal parts to the surface current on the element itself and to the rest of the Universe, each part being $2\pi\mathbf{K} \times \hat{\mathbf{n}}/c$.

The Poynting vector $\mathbf{S}$ from this decomposition has a part due to the surface element along, a part due to the rest of the Universe, and a part due to the cross term. These partial Poynting vectors are $\mathbf{S}/4$, $\mathbf{S}/4$ and $\mathbf{S}/2$, respectively, which is a rather trivial result. No partial Poynting vector has a component perpendicular to the surface of the good/perfect conductor in this decomposition.

A.8 Decompositions for Fields with Arbitrary Time Dependence

Most of the discussion in this note has tacitly assumed sinusoidal time dependence at a single angular frequency ω. In this case, the general form of Poynting theorem,

$$\boldsymbol{\nabla} \cdot \mathbf{S} = -\frac{\partial u}{\partial t} - \mathbf{J} \cdot \mathbf{E}, \tag{20}$$

simplifies, on taking the time average, to,

$$\boldsymbol{\nabla} \cdot \langle \mathbf{S} \rangle = - \langle \mathbf{J} \cdot \mathbf{E} \rangle \,, \tag{21}$$

where the time-average field energy density $\langle u \rangle$, with,

$$u = \frac{E^2 + B^2}{8\pi} \,, \tag{22}$$

(for media with unit relative permittivity and permeability) is a constant at any point in space. A satisfying interpretation of eq. (21) is that only the current density $\mathbf{J}$ acts as a source (or sink) of the time-average flow of energy, $\langle \mathbf{S} \rangle$.

For cases of more general time dependence, the term $-\partial u/\partial t$ in eq. (20) can be thought of as a sink/source of the Poynting vector.[10] This suggests that we might seek decompositions of the fields and of the Poynting vector that have the effect of separating the contributions to the energy flow from the terms $-\mathbf{J} \cdot \mathbf{E}$ and $-\partial u/\partial t$. However, since the various decompositions of the fields considered above have not proved to be satisfactory, it will not be surprising that no plausible decompositions have been suggested to accomplish this. See, for example, [28] for additional commentary on this case.

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