Classical Electromagnetic Fields of a Pair of Annihilating Charges

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1 Problem

The results of annihilation of counterpropagating electrons and positrons has been studied in colliders such as LEP (Large Electron Positron Collider[1]). As the electrons and positrons approach one another, their attraction accelerates them, and they emit (initial-state) radiation called beamstrahlung [2].

Here, we consider an idealized case where the electron and positron approach one another with equal and opposite, constant velocities, until they annihilate at $t = 0$.

The electromagnetic fields of a charge q with uniform velocity $\mathbf{v}$ were first deduced in 1888 by Heaviside [4] (and perhaps more accessibly by Thomson in 1889 [5].

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{\gamma^2 r^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}, \qquad \mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}, \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{1}$$

in Gaussian units, where $\mathbf{r}\mathbf{x} - \mathbf{x}_q$ is the distance from the present position $\mathbf{x}_q(t)$ of the charge to that of the observer at $\mathbf{x}$, ϕ is the angle between $\mathbf{r}$ and $\mathbf{v}$, and c is the speed of light in vacuum.

The form of eq. (1) contrasts with that of the fields of an accelerated chargeas deduced by Liénard (1898) [6] and by Wiechert (1900) [7],

$$\mathbf{E}(\mathbf{x}, t) = q \left[\frac{\hat{\mathbf{r}} - \boldsymbol{\beta}}{\gamma^2 r^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^3} \right] + \frac{q}{c} \left[\frac{\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{r (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \right], \qquad \mathbf{B} = [\hat{\mathbf{r}}] \times \mathbf{E}, \tag{2}$$

where $\boldsymbol{\beta} = \mathbf{v}/c = d\mathbf{x}_q/d\,ct = \dot{\mathbf{x}}_q/c$, $\gamma = 1/\sqrt{1-\beta^2}$, quantities inside brackets [ ] are evaluated at the retarded time, $t_{\rm ret} = t - [r]/c$, $[\mathbf{r}] = \mathbf{x}_q(t) - \mathbf{x}_q(t_{\rm ret})$, and $[\hat{\mathbf{r}}] = [\mathbf{r}]/[r]$. That is, quantities on the right sides of eq (1) are evaluated at the present time of the observer, while those on the right sides of eq. (2) are evaluated at the earlier time consistent with propagation of changes in the fields from the source to the observer at the speed of light.

Because the fields of a uniformly moving charge can be expressed in terms of present quantities, some people infer that the propagation of these fields is instantaneous, as in [8]-[12].

For example, if charges $\pm q$ somehow have uniform velocities $\pm\mathbf{v}$ and collide at the origin at time $t = 0$, then according to eq. (1) the electric field is everywhere zero at that time. Note that the magnetic field nonzero at $t = 0$, being twice that of either charge by itself.

If the charges annihilate one another when they collide, they generate no electromagnetic fields at times $t > 0$.

The claim in [12] is that the electric field is zero everywhere for $t > 0$.

Can this be so?

[1] https://home.cern/science/accelerators/large-electron-positron-collider

2 Solution

If the positive charge has position $\mathbf{x}_+(t) = \mathbf{v}t$ and the negative charge has position $\mathbf{x}_+(t) = -\mathbf{v}t$ for constant velocity $\mathbf{v}$, then according to eq. (1) their electromagnetic fields are,

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_+ + \mathbf{E}_- = \frac{q}{\gamma^2 r_+^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}_+ - \frac{q}{\gamma^2 r_-^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}_-, \tag{3}$$

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_+ + \mathbf{B}_- = \boldsymbol{\beta} \times (\mathbf{E}_+ - \mathbf{E}_-), \tag{4}$$

where $\mathbf{r}_\pm = \mathbf{x} - \mathbf{x}_\pm$.

If the charges somehow pass the through one another at the origin and continued thereafter with uniform velocities, eqs. (3)-(4) would hold for all times.[2]

If instead the charges annihilate at the origin at time $t = 0$, the fields are zero for $t > 0$ inside a sphere of radius ct, and those of eqs. (3)-(4) hold outside it.

In particular, an observer at distance greater than $ct > 0$ from the origin observes a nonzero electric field, with the form as if the charges were still moving with constant velocity (even though the charges ceased to exist at time $t = 0$). The explanation for this by Liénard and Wiechert is that the retarded positions of the charges, according to an observer with $r > ct$, are those as if the charges had not annihilated, but continued to move with uniform velocity.

2.1 Comments

An argument was made on p. 9 of [12] that *the cancelation of the dipole field* (*i.e.*, the annihilation of the two charges) *is an event simultaneous at every point in space and for every observer, regardless of the inertial reference frame in which the observer is at rest.*

This is a serious misunderstanding of the theory of relativity.

It is simply assumed what the paper [12] claims to show, that the effects of annihilation of the charges $\pm q$ propagate instantaneously throughout all space.

Whereas, according to the theory of relativity, while observers in all inertial frames agree that the charges annihilate, say at time t'_a in the $'$ frame, the value of t'_a is different in different frames. More importantly, the relativity of simultaneity means that while an observer at point $P' = \mathbf{x}'$ in the $'$ frame, not at the point of annihilation, agrees that the annihilation took place at time t'_a, an observer in the $''$ frame next to the observer at point P' at time t'_a in the $'$ frame assigns different times to the moment when the annihilation took place, t''_a, and the time t''_o he associates with the observation at $(\mathbf{x}', t'_a)$.

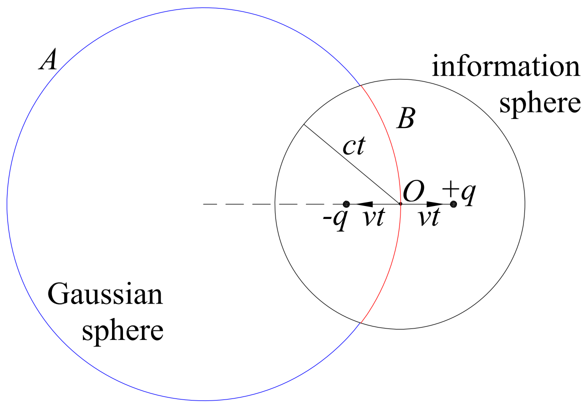
Further, the paper [12] failed to note that if its claim were true, then the magnetic field should vanish everywhere at time $t = 0$ when the charges annihilate, whereas the magnetic field at that time is nonzero everywhere except on the line of motion of the charges. The author of [12] seemed unconcerned with the consequent violation of conservation of energy.

[2] We recall that at time $t = 0$, the electric field (3) is zero everywhere, but the magnetic field (4) is nonzero everywhere, $\mathbf{B}(t = 0) = 2\mathbf{B}_+(t = 0)$. Hence, the energy $U_{\rm EM} = \int (E^2 + B^2)\, d{\rm Vol}/8\pi c$ of the electromagnetic field does not vanish at time $t = 0$, but remains spread out over all space.

The temporary annihilation of the electric field (in the case where the charges pass through one another) could be called a kind of destructive interference. As in other examples of destructive interference (see, for example, [13]), energy is not destroyed by the destructive interference.

A Appendix: A Paradoxical Result

An argument is presented in sec. 3 of [12] based on Gauss' Law, $\oint_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{Area} = 4\pi Q_{\text{inside}}$, whose purpose seems mainly to confuse the reader.



Consider time $t > 0$, with a sphere of radius ct about the origin O, which can be called the information sphere with respect to the collision of the charges $\pm q$ at the origin at $t = 0$, as shown in the figure above. A Gaussian surface is taken as a (different) sphere, whose center is on the line of motion of the two charges, to the left of the origin, with the surface of the sphere including the origin. The Gaussian sphere can be considered to have two parts, A (in blue) and B (in red) in the figure, with part B inside the information sphere.

1. If the charges did not annihilate when they met at the origin at $t = 0$, but passed through one another, the Gaussian sphere contains the negative charge at time $t > 0$, and so the Gaussian integral would be $\oint_1 \mathbf{E} \cdot d\mathbf{Area} = -4\pi q = \int_{1A} + \int_{1B}$, where the electric field $\mathbf{E}$ is due to both uniformly moving charges.

2. If the charges annihilated at the origin at time $t = 0$, the Gaussian sphere contains no charge at time $t > 0$, so the Gaussian integral would be zero, $\oint_2 = 0 = \int_{2A} + \int_{2B}$.

   The electric field on part A, outside the information sphere, is the same whether or not the charges annihilated at the origin, $\int_{1A} = \int_{2A}$. The electric field on part B, inside the information sphere, would be zero if the charges did annihilate at the origin, $\int_{2B} = 0$.

We infer that $0 = \int_{2A} = \int_{1A}$, and hence that $\int_{1B} \mathbf{E} \cdot d\mathbf{Area} = -4\pi q$.

This may seem surprising, but it is consistent with Maxwell's equations.

For another example of Gauss' law in relation to a uniformly moving charge, see the Appendix to [14].

B Appendix: Unequal Velocities (Aug. 18, 2021)

If the charges $\pm q$ meet at the origin at time $t = 0$ with prior, uniform velocities $\mathbf{v}_\pm$ that are not equal and opposite, the story could be obtained by a Lorentz transformation of the case of equal and opposite velocities analyzed above.[3]

[3]The relation is intricate between the general, uniform velocities $\mathbf{v}_\pm$ in the lab frame and the relative velocity $\mathbf{v}$ of the $'$ frame in which the velocities $\mathbf{v}'_\pm$ are equal and opposite.

In this more general case, the electric field is not zero at time $t = 0$, but becomes zero within a sphere of radius ct about the origin at times $t > 0$.

The paradoxical result of Appendix A still holds for the more general case of uniform velocities (which are both less than c in magnitude).

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