What is the Total Angular Velocity of a Rigid Body?

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This question is not answered in textbooks, but we give an answer that can be used with care.

1 Point Mass

We first consider a point mass m that moves with velocity $\mathbf{v}$ when at distance $\mathbf{r}$ from a fixed reference point in an inertial frame.

If is not directed along $\mathbf{r}$, the mass appears to have a rotation with respect to the reference point.

We can write the angular velocity $\boldsymbol{\omega}$ of this rotation as[1]

$$\boldsymbol{\omega} = \frac{\mathbf{r} \times \mathbf{v}}{r^2}\,. \tag{1}$$

The angular momentum $\mathbf{L}$ of the point mass with respect to the reference point is

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}. \tag{2}$$

We can combine these two equations to write

$$\mathbf{L} = mr^2\boldsymbol{\omega} = I\boldsymbol{\omega}, \tag{3}$$

where $I = mr^2$ is the moment of inertia of the point mass with respect to the reference point.

2 Rotating and Translating Rigid Body

Now consider a rigid body of mass m, whose center of mass (c.m.) has position $\mathbf{r}$ and velocity $\mathbf{v}$ with respect to the fixed reference point.

The angular velocity of the c.m. with respect to the reference point is

$$\boldsymbol{\omega}_{\text{of c.m.}} = \frac{\mathbf{r} \times \mathbf{v}}{r^2}\,. \tag{4}$$

The angular momentum of the c.m. motion with respect to the reference point is

$$\mathbf{L}_{\text{of c.m.}} = \mathbf{r} \times m\mathbf{v}. \tag{5}$$

In addition, the body can have a rotation about its c.m., with angular velocity $\boldsymbol{\omega}_{\text{rel to c.m.}}$.[2]

[1] If $r = 0$, we take $\boldsymbol{\omega} = 0$.

[2] In §31 of [1], Landau first calls $\boldsymbol{\omega}_{\text{rel to c.m.}}$ "the angular velocity of rotation", but then says that it can be called "the angular velocity of the body", followed by a cryptic remark that "the velocity of translational motion does not have this absolute property".

If the inertia tensor of the body with respect to its c.m. is $\mathsf{I}_{\rm rel\ to\ c.m.}$, the angular momentum of the body with respect to its c.m. is

$$\mathbf{L}_{\rm\ rel\ to\ c.m.} = \mathsf{I}_{\rm rel\ to\ c.m.} \cdot \boldsymbol{\omega}_{\rm rel\ to\ c.m.}. \tag{6}$$

The total angular momentum of the rotating rigid body, with respect to the reference point, is[3]

$$\mathbf{L}_{\rm\ total} = \mathbf{L}_{\rm\ of\ c.m.} + \mathbf{L}_{\rm\ rel\ to\ c.m.} = mr^2 \boldsymbol{\omega}_{\rm of\ c.m.} + \mathsf{I}_{\rm rel\ to\ c.m.} \cdot \boldsymbol{\omega}_{\rm rel\ to\ c.m.}. \tag{7}$$

And, we can (delicately) say the the total angular velocity of the rigid body with respect to the reference point is

$$\boldsymbol{\omega}_{\rm total} = \boldsymbol{\omega}_{\rm of\ c.m.} + \boldsymbol{\omega}_{\rm rel\ to\ c.m.}, \tag{8}$$

as a kind of mnemonic that there are two components of the angular velocity of a rotating and translating rigid body.

It seems that eq. (8) is not commonly used, perhaps because it appears to involve 2 different reference points. But, note that an observer at the fixed reference point does consider the angular velocity of the rigid body about its c.m. to be $\boldsymbol{\omega}_{\rm rel\ to\ c.m.}$. In this sense, the angular velocity of the body with respect to its c.m. is a "global" property, independent of the choice of a fixed point of reference.[4] Hence, eq. (8) for $\boldsymbol{\omega}_{\rm total}$ is well defined, although $\mathbf{L}_{\rm\ total}$ is not directly related to $\boldsymbol{\omega}_{\rm total}$ for a rigid body.[5]

Thanks to Clement Chan for e-discussions of this topic.

References

[1] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976). https://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics_76.pdf

[2] D. Morin, *Introduction to Mechaincs with Problems and Solutions* (Cambridge U. Press, 2008). https://kirkmcd.princeton.edu/examples/mechanics/morin_07.pdf

[3] A version of our eq. (7) appears as eq. (9.16) of the textbook of Morin [2] without mention of the angular velocity of the center of mass. Note also that in his sec. 9.1.2, Morin tacitly assumed that the center of mass of the rigid body was at rest.

[4] This is discussed in §31 of [1].

[5] A related example of this is the factoid that the velocity of a point at $\mathbf{r}'$ (relative to the fixed reference point) inside the rigid body can be written as $\mathbf{v}_{\mathbf{r}'} = \mathbf{v}_{\rm of\ c.m.} + \boldsymbol{\omega}_{\rm rel\ to\ c.m.} \times (\mathbf{r}' - \mathbf{r})$, which is not of the form $\boldsymbol{\omega}_{\rm total} \times \mathbf{r}'$ unless the c.m. is at rest at the reference point in the lab frame, such that $\mathbf{r} = 0$ and $\mathbf{v}_{\rm of\ c.m.} = 0$.