Breakdown of a Misinterpretation of Noether's Theorem

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
March 23, 2022; updated May 2, 2022)

A recent paper by Lemos [1], titled "Breakdown of the connection between symmetries and conservation laws for semiholonomic systems", leads the reader to suppose that Noether's theorem suffers a "breakdown" in the example of a mass that slides without friction inside a cylinder that rolls without slipping on a horizontal plane.

In classical mechanics, Noether's theorem [2] is a restatement of an insight of Lagrange that if the Lagrangian L of a system is invariant under coordinate q (that is, independent of q), then the CANONICAL (or generalized) momentum $p_q = \partial L/\partial\dot{q}$ is a constant of the motion (*i.e.*, a conserved quantity). Unfortunately, this theorem is often misinterpreted/oversimplified to mean that if the Lagrangian of a system of total mass M is independent of the spatial coordinate x, then the total LINEAR momentum $M\dot{x}$ is conserved. However, the linear momentum is conserved only if $p_x = \partial L/\partial\dot{x} = M\dot{x}$.

In the example of Lemos, the Lagrangian is independent of the horizontal coordinate x of the center of the cylinder, but $p_x = \partial L/\partial\dot{x} \neq M\dot{x}$. Although p_x is a constant of the motion in this example, Lemos's paper suggests that this is a "breakdown" of Noether's theorem because p_x does not equal $M\dot{x}$. This inference is a disservice to Lagrange and to Noether, as a conserved momentum related to an invariance/symmetry does exist in Lemos's example, exactly in accord with Noether's theorem.

Lemos cited Noether's theorem in the Introduction to his paper, and immediately afterwards wrote [1, 3], "The conservation of linear momentum, angular momentum, and energy for many-particle systems is associated with invariance of the action under translations, rotations, and time displacements, respectively." In that context, readers might assume that the quotation *is* Noether's theorem, although it is rather only an important special case thereof. Lemos then went on to show that this statement does *not* hold for his example of the rolling cylinder, because, as is shown in his textbook [4], this statement holds only for "proper" holonomic mechanical systems (as well as for ones with no constraints), but not for semiholonomic systems or for nonholonomic systems [5].

According to the narrow, but popular view of the association/connection between conservation laws and invariance stated in the above quote from Lemos's paper, its title, "Breakdown of the connection between symmetries and conservation laws for semiholonomic systems", is valid. But, the wording of Lemos's paper may lead readers to think that Noether's theorem suffers a breakdown for semiholonomic systems. This note affirms that the full power and validity of Noether's theorem, for any and all systems describable by a Lagrangian, is unaffected by Lemos's result.

References

[1] N.A. Lemos, *Breakdown of the connection between symmetries and conservation laws for semiholonomic systems*, Am. J. Phys. **90**, 221-224 (2022),

http://kirkmcd.princeton.edu/examples/mechanics/lemos_ajp_90_221_22.pdf

[2] See, for example, sec. 13.7 of ref. 1 of Lemos's paper, H. Goldstein, C.P. Poole, Jr., and J.L. Safko, *Classical Mechanics*, 3rd ed. (Addison Wesley, 2001),
http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf
Noether wrote about "invariance" rather than "symmetry" (mainly in the context of general relativity), although the term "symmetry" is now popularly associated with her theorem.
E. Noether, *Invariante Variationsprobleme*, Nachr. König. Gesellsch. Wiss. Göttingen, 235-257 (1918), http://kirkmcd.princeton.edu/examples/mechanics/noether_nkwg_235_18.pdf
http://kirkmcd.princeton.edu/examples/mechanics/noether_nkwg_235_18_english2.pdf

[3] For systems with constraints, the term "action" (used, but not defined in Lemos's paper) can be construed to mean (in the context of Lemos's paper) the "extended" Lagrangian $\mathcal{L}$ defined in his eq. (25), which includes the constraints via a Lagrange multiplier.

[4] The only previous demonstration of this fact may be that on p. 69 of Lemos's book, *Analytical Mechanics* (Cambridge U.P., 2018),
http://kirkmcd.princeton.edu/examples/mechanics/lemos_18.pdf

[5] The terms "holonomic" and "semiholonomic" are discussed on p. 264 of ref. 9 of Lemos's paper, J.G. Papastavridis, *Analytical Mechanics* (World Scientific, 2014),
http://kirkmcd.princeton.edu/examples/mechanics/papastravridis_02.pdf
"Holonomic" systems have constraints of the form $f_i(\{q_j\}, t) = 0$. "Semiholonomic" systems have velocity-dependent constraints of the form $g_i(\{q_j, \dot{q}_j\}, t) = 0$ that can be integrated to the holonomic form, but which include constants that depend on the initial conditions. Hence, semiholonomic systems are a subset of holonomic ones. A distinct subset of holonomic systems is sometimes called "proper," for which constants in the constraints of these holonomic subsystems are independent of the initial conditions.

The term "holonomous" = integral (ὅλος) laws ($\nu o \mu \acute{o} \varsigma$) was introduced by H. Hertz in sec. 123, p. 80 of *The Principles of Mechanics* (Macmillan, 1899),
http://kirkmcd.princeton.edu/examples/mechanics/hertz_99.pdf
See also p. 91 of the original German edition (Barth, 1894),
http://kirkmcd.princeton.edu/examples/mechanics/hertz_mechanik_94.pdf