Abbott's Puzzler

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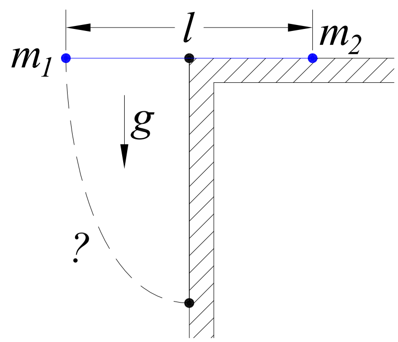
1 Problem

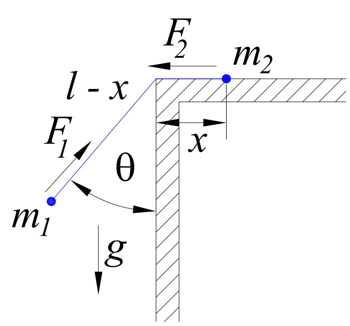
Two "point" masses m_1 and m_2 are connected by a "massless" string of fixed length l. The system is initially horizontal and at rest, with m_2 on a frictionless horizontal surface, and the midpoint of the string at the edge of the horizontal surface, as sketched below. For what ratio m_1/m_2 does mass m_1 (after it is released) strike the vertical surface below the edge of the horizontal surface at the same time that mass m_2 slides off the horizontal surface to the left, assuming the string is straight at all times between mass m_1 and the edge of the horizontal surface?



This problem was posed by Derek Abbott.

2 Solution

If $m_2 \gg m_1$, then mass m_2 barely moves while mass m_1 falls and hits the vertical surface. On the other hand, if $m_2 \ll m_1$, then mass m_1 falls approximately vertically while mass m_2 slides to the left, and off the horizontal surface before mass m_1 hits the vertical surface. We infer that there exists some value of the ratio m_1/m_2 such that mass m_1 hits the vertical surface at the moment when mass m_2 reaches the edge of the horizontal surface.



If we can ignore friction, the energy $E + T + V$ is conserved, but this is the only conserved quantity, while there are two degrees of freedom in this problem, taken as coordinates x and θ in the figure above.

The kinetic energy is

$$T = \frac{m_1}{2}\left(\dot{x}^2 + (l-x)^2\dot{\theta}^2\right) + \frac{m_2\dot{x}^2}{2}, \tag{1}$$

and the gravitational potential energy can be taken as

$$V = -m_1 g(l-x)\cos\theta. \tag{2}$$

From the Lagrangian $L = T - V$, the two equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m_1(l-x)^2\,\ddot{\theta} - 2m_1(l-x)\dot{x}\,\dot{\theta} = \frac{\partial L}{\partial\theta} = -m_2 g(l-x)\sin\theta, \tag{3}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = (m_1+m_2)\ddot{x} = \frac{\partial L}{\partial x} = -m_1(l-x)\,\dot{\theta}^2 - m_1 g\cos\theta. \tag{4}$$

The total energy can be taken as zero,

$$E = T + V = 0 = \frac{m_1}{2}\left(\dot{x}^2 + (l-x)^2\dot{\theta}^2\right) + \frac{m_2\dot{x}^2}{2} - m_1 g(l-x)\cos\theta, \tag{5}$$

but this does not lead to any significant simplification of either of the equations of motion (3)-(4). It seems that numerical investigation of these equations is needed to determine the ratio m_1/m_2 for which mass m_1 strikes the vertical plane just as mass m_2 reaches the edge of the horizontal plane.

In Lagrange's method, the tension in the string between the two masses is not considered. We obtained the equations of motion (3)-(4) without the need to clarify whether the tension F_2 to the left in the horizontal portion of the string,

$$F_2 = m_2\,a_{\rm horiz} = -m_2\ddot{x}, \tag{6}$$

is the same as the tension F_1 in the string between mass m_1 and the edge of the horizontal surface, which is related by

$$m_1 a_{\rm along\ string} = m_1\left(-\ddot{(l-x)} + (l-x)\dot{\theta}^2\right) = m_1\left(\ddot{x} + (l-x)\dot{\theta}^2\right) = F_1 - m_1 g\cos\theta. \tag{7}$$

It might be considered as "intuitively obvious" that $F_1 = F_2$ in magnitude, although the edge of the horizontal surface exerts a force $\mathbf{F}_3$ on the string such that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$, which does not require that $F_1 = |\mathbf{F}_1| = F_2 = |\mathbf{F}_2|$. Anyway, if we suppose that $F_1 = F_2 = F$, then combining eqs. (6) and (7) leads to the Lagrangian result (4), which confirms this supposition is valid within the approximations assumed here.

For completeness, we note that the acceleration of mass 1 perpendicular to the string (and in the direction of increasing θ) is related by

$$m_1 a_\perp = m_1(l-x)\,\ddot{\theta} - 2m_1\dot{x}\,\dot{\theta} = -m_1 g\sin\theta, \tag{8}$$

which is the equation of motion (3) divided by $l - x$. *Thus, a Newtonian analysis suffices provided we accept that* $F_1 = F_2 = F$.

The horizontal acceleration of mass 2 has magnitude $a_{2h} = F/m_2$, *while the horizontal acceleration of mass 1 has magnitude* $a_{1h} = F \sin\theta/m_1$. *If* $m_1 = m_2$ *then* $a_{1h} < a_{2h}$, *so the time to travel horizontal distance* $l/2$ *is less for mass 2 and mass 1, and mass 2 slides off the horizontal surface before mass 1 strikes the vertical surface. For both masses to travel horizontal distance* $l/2$ *in the same time, we must have* $m_1 < m_2$. *As noted above, if* $m_1 \ll m_2$, *then* m_1 *strikes the vertical surface before* m_2 *has moved very far. Apparently the solution to Abbott's puzzler is that* m_1 *is "somewhat" less than* m_2.