Significance of the Vector Potential

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In 1831, Faraday [1] intuited aspects of the electromagnetic vector potential $\mathbf{A}$, which he called the electronic state, and since then attitudes as to the significance of the vector potential have varied considerably.[1]

Maxwell (1864) discussed the "electromagnetic momentum of a current" in sec. 22 of [6], and identified this in sec. 59 with what he later called the vector potential.[2] In secs. 98-99 of [6], Maxwell gave the first discussion of what is now called a gauge transformation, which he noted can change the form of the vector potential. To maintain the interpretation of the vector potential as an "electromagnetic momentum", Maxwell argued that the vector potential should obey $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, which is now called the Coulomb-gauge condition.

Maxwell (1873) returned to this theme in Arts. 616-617 of [5] where he claimed that the vector potential has the form $\mathbf{A} = \int \mathbf{J}\, dVol/cr$ (in Gaussian units) where c is the speed of light. We now recognize this as a possible form of the vector potential only if the electric-current density $\mathbf{J}$ is static (*i.e.*, $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$). Maxwell did not live long enough to reflect on the conundrum that his vision of the vector potential as an "electromagnetic momentum" applied only to static examples, in which the total momentum should be zero, so if there is a nonzero "electromagnetic momentum" there must also be an equal and opposite "hidden" mechanical momemtum.[3]

Heaviside, one of Maxwell's immediate followers, was skeptical of the significance of the vector potential, and stated in 1886 [9], "the very artificial nature of $\mathbf{A}$ and P (*the electric scalar potential*) greatly obscures and complicates many investigations". In 1889 [10] he added that "the fact that $\mathbf{A}$ has often a scalar potential parasite ... causes sometimes great mathematical complexity and indistinctness ; and it is, for practical reasons, best to murder the whole lot, or at any rate merely employ them as subsidiary functions."

In 1890, Hertz [11] characterized the electromagnetic potentials as "scaffolding ... which serve for calculation only".

In 1900 Poincaré [12] argued that the volume density $\mathbf{p}_{\rm field}$ of electromagnetic-field momentum is the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ (in SI units) divided by c^2, where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields and c is the speed of light.[4] Then, the electromagnetic-field momentum is related by $\mathbf{P}_{\rm field} = \int \mathbf{p}_{\rm field}\, d{\rm Vol} = \epsilon_0 \int \mathbf{E} \times \mathbf{B}\, d{\rm Vol}$ in SI units (with permittivity ϵ_0 replaced by $1/4\pi c$ in Gaussian units, which will be used below).

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[1]Gauss, p. 609 of [2] (published posthumously in 1867), claimed to have invented a version of the vector potential in 1835. The vector potential is often attributed to Neumann (1845) [3], but he only considered a scalar magnetic-field (potential) energy rather than a vector potential. The earliest published example of a vector potential is due to Thomson (1846), p. 63 of [4]. The first published use of the term "vector potential" is in Art. 405 of Maxwell's *Treatise* (1873) [5].

[2]In contrast, Maxwell stated (1881) [7] "The electrical potential ... is a mere scientific concept. We have no reason to regard it as denoting a physical state."

[3]It is now recognized that the "hidden" mechanical momentum is a relativistic effect associated with the current density $\mathbf{J}$. See, for example [8].

[4]For a brief history of this insight, see p. 246 of [13].

The electromagnetic-field momentum is a global property of the electromagnetic fields and depends on the details of their distant sources.

In general, electromagnetic-field momentum is not associated with the vector potential, although in static examples the electromagnetic-field momentum can also be computed (following Maxwell, and in Gaussian units) as $\mathbf{P}_{\text{field}} = \int \rho \mathbf{A}^{(\text{C})} \, d\text{Vol}/c$ where $\mathbf{A}^{(\text{C})}(\mathbf{x}) = \int \mathbf{J}(\mathbf{x}') \, d\text{Vol}/cr$ is the vector potential in the Coulomb gauge supposing that $\mathbf{A}^{(\text{C})} = 0$ at infinity, ρ is the electric-charge density, $\mathbf{J}$ is the electric-current density and $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. This reinforces that the vector potential is "useful" and "valid", but it does not imply that the vector potential is "physical", as suggested in [14]-[22].

In static examples the electromagnetic-field momentum can be computed in other ways as well. The form $\mathbf{P}_{\text{field}} = \int V^{(\text{C})} \mathbf{J} \, d\text{Vol}/c^2$, where $V^{(\text{C})} = \int \rho \, d\text{Vol}/r$ is the electric scalar potential in the Coulomb gauge was first advocated Furry [23]. And, the form $\mathbf{P}_{\text{field}} = \int (\mathbf{J} \cdot \mathbf{E}) \, \mathbf{r} \, d\text{Vol}/c^2$ where $\mathbf{E} = \int \rho \, \hat{\mathbf{r}} \, d\text{Vol}/r^2$ was introduced by Aharonov, Pearle and Vaidman [24].

The density of momentum in the electromagnetic field is $\mathbf{p}_{\text{field}} = \mathbf{S}/c^2 = \mathbf{E} \times \mathbf{B}/4\pi c$, and not $\rho \mathbf{A}^{(\text{C})}/c$ nor $V^{(\text{C})} \mathbf{J}/c^2$ nor $(\mathbf{J} \cdot \mathbf{E}) \, \mathbf{r}/c^2$.

Recall that the canonical momentum of an electric charge q with (rest) mass m and velocity $\mathbf{v}$ in an electromagnetic field can be written as $\mathbf{p}_{\text{canon}} = \mathbf{p}_{\text{mech}} + q\mathbf{A}/c$ for the vector potential $\mathbf{A}$ in any gauge, with $\mathbf{p}_{\text{mech}} = m\mathbf{v}/\sqrt{1 - v^2/c^2}$. Then, the Hamiltonian, $H = c[m^2c^2 + (\mathbf{p}_{\text{canon}} - q\mathbf{A}/c)^2]^{1/2} + qV$, of the electric charge in an electromagnetic field appears not to be gauge invariant, but the equation of motion, obtained by taking derivatives of the Hamiltonian, is gauge invariant, being $m \, \mathbf{a} = q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$, *i.e.*, the Lorentz force law. This is perhaps the most significant descendant of Maxwell's notion that the vector potential is a kind of "electromagnetic momentum".

Maxwell's partial understanding of the gauge invariance of the electromagnetic potentials was made more explicit by Lorentz in 1905, p. 157 of [25]. See also p. 673 of [26]. That is, the physical fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from potentials $\mathbf{A}$ and V according to $\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A}/\partial ct$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, while the nonphysical potentials can be changed arbitrarily, with no effect on $\mathbf{E}$ or $\mathbf{B}$, according to $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$ and $V \to V + \partial\chi/\partial ct$ for any (doubly) differentiable function χ.

In quantum theory, where the Hamiltonian plays a key role in Schrödinger's equation, the appearance of the electromagnetic potentials in the Hamiltonian of a charged particle in an electromagnetic field gives indicates that the unphysical potentials are nonetheless "fundamental".[5]

In 1926, Fock noted [30, 31, 32] that Schrödinger's equation for an electric charge in an electromagnetic field is gauge invariant only if the gauge transformation of the potentials, $A_\mu(x_\nu) \to A_\mu + \partial_\mu \chi(x_\nu)$, is accompanied by a phase change of the quantum wavefunction, $\psi(x_\nu) \to e^{-ie\chi(x_\nu)/\hbar c} \, \psi$. Yang and Mills (1954) [33, 34] were the first to point out that Fock's argument can be inverted such that a requirement of local phase invariance of the form

[5]Because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from knowledge of the electromagnetic potentials V and $\mathbf{A}$, the latter are often considered to be more "fundamental" than the former. However, the potentials V and $\mathbf{A}$ can be deduced from the fields $\mathbf{E}$ and $\mathbf{B}$ via an argument attributed to Poincaré [27], so it is not self evident which is more "fundamental" than the other. Consideration of the vector potential in the Poincaré gauge [28] for the Aharanov-Bohm effect [29] indicates that one should not suppose the vector potential always provides a "local" explanation of this effect.

$\psi(x_\nu) \to e^{-ie\chi(x_\nu)/\hbar c}\,\psi$ implies the existence of an interaction described by a 4-potential A_μ which satisfies gauge invariance, namely the electromagnetic interaction.[6] This emphasizes the significance of potentials in the quantum realm, and has led to the understanding of the electroweak and strong interactions as further examples of "gauge theories".[7]

It can remain unsettling that nonphysical potentials play a key role in determining what type of physical fields are possible in Nature.

References

[1] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125-162 (1832). See Art. 60. https://www.jstor.org/stable/107956
kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[2] C.F. Gauss, *Werke*, Vol. 5 (Göttingen, 1867).
https://archive.org/details/Werkecarlf05gausrich/page/n620/mode/1up
kirkmcd.princeton.edu/examples/EM/gauss_werke_v5.pdf

[3] F.E. Neumann, *Allgemeine Gesetze der inducirten elektrischen Ströme*, Abh. König. Akad. Wiss. Berlin, pp. 1-87 (1845).
https://archive.org/details/diemathematische00neumuoft
kirkmcd.princeton.edu/examples/EM/neumann_akawb_1_45.pdf

[4] W. Thomson, *On the Mathematical Theory of Electricity in Equilibrium*, Cam. Dublin Math. J. **1**, 75 (1846). https://babel.hathitrust.org/cgi/pt?id=cub.u183019465063&seq=83
kirkmcd.princeton.edu/examples/EM/thomson_cdmj_1_75_46.pdf

[5] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, Cambridge, UK, 1873). https://archive.org/details/electricandmagne02maxwrich
http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[6] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459-512 (1865). doi/10.1098/rstl.1865.0008
kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[7] J.C. Maxwell, *An Elementary Treatise on Electricity* (Clarendon Press, 1881). See p. 53.
https://archive.org/details/elementarytreati00maxwrich/page/56/mode/1up
http://kirkmcd.princeton.edu/examples/EM/maxwell_elementary_treatise_81.pdf

[8] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826-833 (2009). See p. 828. https://doi.org/10.1119/1.3152712
http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf

[6] Note that an implication of local phase invariance of the quantum wavefunction ψ is that the wave function ψ (like the potentials) is not physical.

[7] See, for example, sec. I of [26].

[9] O. Heaviside, *On the Self-induction of Wires*, Phil. Mag. **22**, 118-138 (1886). See p. 124.
https://doi.org/10.1080/14786448608627908
https://kirkmcd.princeton.edu/examples/EM/heaviside_pm_22_118_86.pdf

[10] O. Heaviside, *The general solution of Maxwell's electromagnetic equations in a homogeneous isotropic medium, especially in regard to the derivation of special solutions, and the formulæ for plane waves*, Phil. Mag. **27**, 29-50 (1889). See p. 45.
https://doi.org/10.1080/14786448908628314
https://kirkmcd.princeton.edu/examples/EM/heaviside_pm_22_118_86.pdf

[11] H. Hertz, *Ueber die Grundgleichungen der Electrodynamik für ruhende Körper*, Ann. d. Phys. **40**, 577-624 (1890). See p. 578. https://doi.org/10.1002/andp.18902760803
https://kirkmcd.princeton.edu/examples/EM/hertz_ap_40_577_90.pdf
Translation in H. Hertz, *Electric Waves* (Macmillan, 1900). See p. 107.
https://archive.org/details/electricwavesbe00jonegoog/page/n219/mode/1up
http://kirkmcd.princeton.edu/examples/EM/hertz_electric_waves.pdf

[12] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900). https://henripoincarepapers.univ-lorraine.fr/chp/hp-pdf/hp1900an.pdf
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
English translation: http://physicsinsights.org/poincare-1900.pdf
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf
Poincaré's paper deals with macroscopic electrodynamics, in which $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ in SI units with $\mathbf{E} = (f, g, h)$, $\mathbf{B} = (\alpha, \beta, \gamma)$. The equations on p. 6 before eqs. (4) state that the electromagnetic-field-energy density u ($= K_0 J$) times the energy flow velocity $\mathbf{U}$ equals the Poynting vector $\mathbf{S}$, *i.e.*, $u\mathbf{U} = \mathbf{S}$. Just after his eq. (7) Poincaré stated that the field-energy density u equals the effective mass density $\rho_{\rm eff}$ times c^2, *i.e.*, $u = \rho_{\rm eff}\, c^2$. Then, we have $\rho_{\rm eff}\mathbf{U}c^2 = \mathbf{g}\, c^2 = \mathbf{S}$, where $\mathbf{g} = \rho_{\rm eff}\mathbf{U}$ is the field-momentum density.

[13] K.T. McDonald, *The Poynting vector should be* $\mathbf{E} \times \mathbf{B}/\mu_0$ *not* $\mathbf{E} \times \mathbf{H}$, Am. J. Phys. **94**, 245-247 (2026). https://doi.org/10.1119/5.0297408
http://kirkmcd.princeton.edu/examples/EM/mcdonald_ajp_94_245_26.pdf

[14] E.J. Konopinski, *What the electromagnetic vector potential describes*, Am. J. Phys. **46**, 499-502 (1978). https://doi.org/10.1119/1.11298
http://kirkmcd.princeton.edu/examples/EM/konopinski_ajp_46_499_78.pdf

[15] M.D. Semon and J.R. Taylor, *Thoughts on the magnetic vector potential*, Am. J. Phys. **64**, 1361-1369 (1996). https://doi.org/10.1119/1.18400
http://kirkmcd.princeton.edu/examples/EM/semon_ajp_64_1361_96.pdf

[16] N.K. Georgieva and H.W. Tam, *Potential Formalisms in Electromagnetic-Field Analysis*, IEEE Trans. Micro. Tech. Meas. **51**, 1330-1338 (2003).
https://doi.org/10.1109/TMTT.2003.809188
https://kirkmcd.princeton.edu/examples/EM/georgieva_ieeemtt_51_1330_03.pdf

[17] G. Rousseaux, *On the physical meaning of the gauge conditions of Classical Electromagnetism: the hydrodynamics analogue viewpoint*, Ann. Fond. Louis de Broglie **28**,

261-269 (2003). https://hal.science/hal-00013234
https://kirkmcd.princeton.edu/examples/EM/rousseaux_aflb_28_261_03.pdf

[18] D. Iencinella and G. Matteucci, *An introduction to the vector potential*, Eur. J. Phys. **25**, 249-256 (2004). https://doi.org/10.1088/0143-0807/25/2/011
https://kirkmcd.princeton.edu/examples/EM/iencinella_ejp_25_249_04

[19] G. Rousseaux, R. Kofman and O. Minazzoli, *The Maxwell-Lodge effect: Significance of electromagnetic potentials in the classical theory*, Eur. Phys. J. D **49**, 249-256 (2008).
https://doi.org/10.1140/epjd/e2008-00142-y
https://kirkmcd.princeton.edu/examples/EM/rousseaux_epjd_49_249_08.pdf

[20] G. Giuliani, *Vector potential, electromagnetic induction and 'physical meaning'*, Eur. J. Phys. **31**, 871-880 (2010). http://dx.doi.org/10.1088/0143-0807/31/4/017
https://kirkmcd.princeton.edu/examples/EM/giuliani_ejp_31_871_10.pdf

[21] S. Barbieri, M. Cavinato and M. Giliberti, *An educational path for the magnetic vector potential and its physical implications*, Eur. J. Phys. **34**, 1209-1219 (2013).
http://dx.doi.org/10.1088/0143-0807/34/5/1209
https://kirkmcd.princeton.edu/examples/EM/barbieri_ejp_34_1209_13.pdf

[22] J. Adams and C. Dunston, *Numerical modeling of the magnetic vector potential and the associated electromagnetic momenta*, Am. J. Phys. **94**, 113-119 (2026).
https://doi.org/10.1119/5.0246553
http://kirkmcd.princeton.edu/examples/EM/adams_ajp_94_113_26.pdf

[23] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621-636 (1969).
https://doi.org/10.1119/1.1975729
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[24] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052-4055 (1988). https://doi.org/10.1103/PhysRevA.37.4052
http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf

[25] H.A. Lorentz, *Weiterbildung der Maxwellischen Theorie. Elektronentheorie*, in *Encyklopädie der Mathematischen Wissenschaften*, Band 5, Teil 2 (1904), pp. 145–280.
https://kirkmcd.princeton.edu/examples/EM/lorentz_emw_5_2_145_04.pdf

[26] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663-680 (2001). https://doi.org/10.1103/RevModPhys.73.663
https://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73

[27] W.E. Brittin, W.R. Smythe and W. Wyss, *Poincaré gauge in electrodynamics*, Am. J. Phys. **50**, 693-696 (1982). https://doi.org/10.1119/1.12731
https://kirkmcd.princeton.edu/examples/EM/brittin_ajp_50_693_82.pdf

[28] K.T. McDonald, *Vector Potential of a Long Solenoid in the Poincaré Gauge* (January 15, 2017). https://kirkmcd.princeton.edu/examples/poincare.pdf

[29] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485-491 (1959). https://doi.org/10.1103/PhysRev.115.485
http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf

[30] V. Fock, *Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt*, Z. Phys. **39**, 226-232 (1926).
https://doi.org/10.1007/BF01321989
http://kirkmcd.princeton.edu/examples/QM/fock_zp_39_226_26.pdf
http://kirkmcd.princeton.edu/examples/QM/fock_zp_39_226_26_english.pdf

[31] See also F. London, *Quantenmechanische Deutung der Theorie von Weyl*, Z. Phys. **42**, 375-389 (1927). https://doi.org/10.1007/BF01397316
http://kirkmcd.princeton.edu/examples/QM/london_zp_42_375_27.pdf

[32] See also, p. 206 of W. Pauli, *Relativistic Field Theories of Elementary Particles*, Rev. Mod. Phys. **13**, 203-232 (1941). https://doi.org/10.1103/RevModPhys.13.203
http://kirkmcd.princeton.edu/examples/EP/pauli_rmp_13_203_41.pdf

[33] C.N. Yang and R.L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96**, 191-195 (1954). https://doi.org/10.1103/PhysRev.96.191
http://kirkmcd.princeton.edu/examples/EP/yang_pr_96_191_54.pdf

[34] See also R. Utiyama, *Invariant Theoretical Interpretation of Interaction*, Phys. Rev. **101**, 1597-1607 (1956). https://doi.org/10.1103/PhysRev.101.1597
http://kirkmcd.princeton.edu/examples/EP/utiyama_pr_101_1597_56.pdf