CREATION AND ANNIHILATION OPERATORS FOR ELECTRONS

Up to this point we have been using a formalism in which bosons (photons, phonons, etc.) are described by creation and annihilation operators, while the electrons in the system have been treated by standard wave function methods. One might expect that a certain elegance would result from use of a formalism which also contained creation and annihilation operators for fermions. However, the main reason for applying this language to electrons is not elegance, but rather the fact that certain phenomena, such as pair creation, seem much more reasonable when approached in this manner.

What we want is some sort of operator $c$ such that $c_{k,i}^+$ creates an electron with momentum $k$ in spin state $i$, and $c_{k,i}$ removes such an electron from the picture.

(To simplify things, the spin index will be left out; any student who is confused by this should go through and reinsert all the indices.) If we had such an operator, we could write down interaction Hamiltonians which would in first order produce any desired process.

Example:

Electron $k$ scatters to electron $k'$ by absorbing phonon $\hbar$.

This is produced by an interaction like

$$\sum \mathcal{P}_{k,k',k} \mathcal{R}_{k,k'} c_{k,k',k}^+ c_{k'}^+ c_{k'} c_{k}$$

However if this is part of a Hamiltonian operator, its Hermitian conjugate must be included also

$$\mathcal{H}^{\text{end}} = \sum \mathcal{P}_{k,k',k} \mathcal{R}_{k,k'} c_{k,k',k}^+ c_{k'}^+ c_{k'} c_{k} + \sum \mathcal{P}_{k',k,k} \mathcal{R}_{k,k'} c_{k,k',k}^+ c_{k'}^+ c_{k'} c_{k}$$

Knowledge of the $\mathcal{P}$'s would then allow perturbation theory calculations of all sorts of processes, with $\mathcal{H}^{\text{end}}$ as perturbation.

To perform such calculations, one must know the properties of these $c^s_{k,i}$ as operators.

To begin with, suppose a universe of one electron state. It can either be empty or filled with one electron. Hence we arrive at

$$c^s |1\rangle = |1\rangle$$

$$c^s |0\rangle = 0$$

By taking the adjoint,

$$c |1\rangle = |0\rangle$$

$$c^* c = 0$$
\[ c|0\rangle = 0 \]

Similarly \( c c^\dagger + c^\dagger c = 1 \)

(test by applying to all possible states)

and \( c_i^\dagger c_i = M_i \) (where \( M_i \) is either 0 or 1)

Puzzle: Suppose you know only that an operator \( c \) obeys

\[ c^\dagger c^\dagger = 0 \quad c c = 0 \quad c c^\dagger + c^\dagger c = 1 \]

Find a representation for the operator. Is it unique?

When the universe is expanded to include two electrons, a little bookkeeping becomes necessary. Let electron \#1 be the one which is created first.

\[ c_a^\dagger c_b^\dagger |0\rangle = |a b\rangle \]

means electron \#1 is in state \( b \); electron \#2 is in state \( a \).

The electron wave functions must be antisymmetric under interchange of electrons 1 and 2

\[ |a b\rangle = -|b a\rangle \implies c_a^\dagger c_b^\dagger |0\rangle = -c_b^\dagger c_a^\dagger |0\rangle \]

In fact the same relation holds for states other than the vacuum and we have the operator equations

\[ c_a^\dagger c_b^\dagger = -c_b^\dagger c_a^\dagger \]

\[ c_a c_b = -c_b c_a \]

The next thing to check is clearly \( c_a c_b^\dagger = -c_b c_a^\dagger \)

There is a clever way to determine this:

The equation \( c_c^\dagger c_c + c_c^\dagger c_c^\dagger = 1 \) must hold for the creation operator for any electron state. By the principle of superposition, if \( |a\rangle \) and \( |b\rangle \) are states, a perfectly allowable state is \( \alpha |a\rangle + \beta |b\rangle = |c\rangle \) \( \exists |\alpha|^2 + |\beta|^2 = 1 \)

\[ c_c^\dagger = \alpha c_a^\dagger + \beta c_b^\dagger \]

Hence

\[ (\alpha c_a^\dagger + \beta c_b^\dagger)(\alpha^* c_a + \beta^* c_b) + c_a^\dagger c_b^\dagger (\alpha c_a^\dagger + \beta c_b^\dagger) = 1 \]

Expand. Use \( |\alpha|^2 + |\beta|^2 = 1 \)

Then

\[ \beta c_a^\dagger c_b c_a + \alpha^* \beta^* c_b^\dagger c_a + \beta^* \alpha c_b c_a^\dagger + \alpha^* \beta^* c_b c_a^\dagger = 0 \]

\[ c_b^\dagger c_a + c_a c_b^\dagger = 0 \]
We thus have a set of anticommutation relations for fermions which parallel the commutation relations previously derived for bosons:

\[ [c_a, c_b^+] = \delta_{ab} \quad \text{if } a \neq b \]

\[ [c_a^+, c_b^+] = 0 \]

Exercise: figure out what these relations are if \( a \) and \( b \) are not orthogonal.

Above we determined the operator \( c_\alpha^+ = \alpha c_\alpha^+ + \beta c_\beta^+ \) which creates an electron with amplitude \( \alpha \) to be in state \( a \) and amplitude \( \beta \) to be in state \( b \). Likewise, \( c_\alpha^+ = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \) creates an electron with amplitude \( \langle \rho | \alpha \rangle \) to have momentum \( \rho \).

This means its wave function must have the form \( \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} = \hat{\chi}(x) \)

And \( c_\alpha^+ = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \). If we symbolize the creation operator by \( \langle \alpha | \chi \rangle \)

\[ \chi^+(x_0) \]

then \( \chi^+(x_0) = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \).

This sort of formalism makes it obvious that the manipulations performed here with operators are the same as usually performed with the states they create.

Let \( \chi^+(x_0) \) be an operator which creates an electron at \( x_0 \). Then \( \chi^+(x_0) = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \)

\[ \chi^+(x) = \int d^3 \rho \cdot \alpha \cdot \chi \]

Exercise: using the properties of the \( e^{i \rho \cdot \alpha} \), convince yourself that

\[ \chi^+(x) \chi^-(y) - \chi^-(y) \chi^+(x) = \delta^3(x-y) \]

These \( \chi^+(x) \) operators are called field operators. Because \( \chi^+(x) \)

and \( c_\alpha^+ = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \)

Then \( C^+ = \int \frac{\chi^+(x)}{f_{\text{wave} \, f_a}} d^3 x \)

The field operator may be thought of as just a device which changes wave functions into the corresponding operators.

These methods can equally well be applied to solid state physics. If \( f_\mu (x) = e^{i k \cdot x} u_\mu (x) \)

is the wave function for an electron of propagation vector \( k \), then the operator which creates such a particle is \( \hat{\chi} = \sum \langle \rho | \alpha \rangle e^{i \rho \cdot \alpha} \)

Thus, for an arbitrary number of non-interacting electrons, the Hamiltonian of the system can be written as

\[ H = \sum \varepsilon_k n_k = \sum \varepsilon_k \chi^+_k \chi_k \]

But \( \chi^+_k = \sum \chi^+_k \)

and if \( h \) is the Hamiltonian for the individual electron wave function, then

\[ h \chi^+_k = \sum \chi^+_k \]

\[ \chi^+_k \]
\[ \int \psi^* \psi \, d^3x = \int d^3x \sum_K \zeta_K \chi_K^* \chi_K \int \psi^* \psi \, d^3x \]

Thus the Hamiltonian operator for the entire system can be expressed in terms of the Hamiltonian for an individual wave function and the field operator by

\[ \hat{H} = \int \psi^* \psi \, d^3x \]

Convince yourself that the creation operators for electrons must commute with all the operators for phonons. Then it is easy to see that a sample Hamiltonian for an interacting system of phonons and electrons might be written

\[ \hat{H} = \sum_K \varepsilon_K \chi_K^* \chi_K + \sum_K \hbar \omega_k \alpha_K^* \alpha_K \]

\[ + \sum_{K' K} [ \chi_K^* \chi_K \alpha_K \beta_{K' K} + \chi_K^* \chi_K \beta_{K' K} \alpha_{K'}^* ] \]

This formalism can be used to treat any system in which electrons interact only by exchange of phonons. If, in addition to this, there is some direct electron-electron interaction, a further development is necessary.

If the electrons interact through some potential \( V(r_1, r_2) \) then the amplitude for scattering by this potential will be proportional to \( \psi(Q) = \int V(r) e^{-iQ \cdot r} \, d^3r \)

Then the definitions of the creation operators imply that the Hamiltonian should contain terms of the form

\[ \sum_{\rho} \left( \frac{\hbar^2}{2m} \right) c_{\rho}^* c_{\rho} + \sum \frac{\hbar^2}{2m} \sum \langle \rho, \sigma \rangle \delta(\rho_1^* \rho_2 - \rho_3 - \rho_4) c_{\rho_1}^* c_{\rho_2}^* c_{\rho_3} c_{\rho_4} \]

Use of the relation \( c_{\rho}^* = \int \psi^* \chi(x) \, dx \)

allows us to rewrite the interaction term as

\[ \int \psi^*(x) \psi^*(y) V(x-y) \psi(x) \psi(y) \, d^3x \, d^3y = \int \psi^*(x) \psi(x) V(0) \, d^3x \]

To understand this, we compare it with the classical expression for interaction of two charge densities.

\[ G_{\text{classical}} = \frac{1}{2} \int \rho(x) \rho(y) \, d^3x \, d^3y \]

By doing this we discover:

a) we should have defined the interaction as \( \frac{1}{2} \sum V(Q) \sum c_{\rho}^* c_{\rho} c_{\rho} c_{\rho} \)

b) there seems to be an extra term \( \int \psi^*(x) \psi(x) V(0) \, d^3x \)

This means that our quantum interaction contains no self-energy. This is exactly what we want (especially for things like a Coulomb potential where the self-energy
is infinite.

c) There is a difference of a sign between the classical result and the one we found. But we can fix this just by permuting the fermion creation operators in the definition. So this is not serious or important.

Hence the Hamiltonian for a system of any number of electrons, interacting through the Coulomb potential, can be expressed as

\[
H = \int \Psi^+(x) \left[ -\frac{\hbar^2}{2m} \left( \nabla - eA(x,t) \right)^2 + e\phi(x,t) \right] \Psi(x) \, d^3x \\
+ \frac{1}{2} \int \int \Psi^+(x) \Psi^+(y) \frac{e^2}{|x-y|^2} \Psi(y) \Psi(x) \, d^3x \, d^3y
\]

where the \( \Psi \)s are field operators.

With this Hamiltonian, the Schrödinger equation for such a system is

\[
-\frac{i}{\hbar} \frac{\partial \Psi}{\partial t} = H \Psi
\]

Notice that there is no reason for the solution \( \Psi \) to be composed of a definite number of electrons. However, because this particular Hamiltonian commutes with the particle number operator \( \mathcal{N} = \int \Psi^+(x) \Psi(x) \, d^3x \)

it is possible to choose solutions which do have the property \( \mathcal{N} \Psi = n \Psi \)

In general, this interaction will not mix subspaces with different \( n \).
Let $C_k^*$ be the operator that creates an electron in state $k$. Then $\sum_k C_k C_k^* \frac{\gamma}{2} = \delta_{kk'}$ and $\sum_k C_k C_k^* \frac{\gamma}{2} = 0$.

A simple Hamiltonian that permits discussion of electron scattering in a crystal is

$$H = \sum_k \varepsilon_k C_k^* C_k + \sum_{k<k'} \lambda_{kk'} C_k^* C_{k'}$$

An unfilled electron state in an energy band is called a "hole." Electron-creation operators are hole destruction operators and vice versa.

The formal identification is $C_k = b_k^*$, consequently

$$\sum_{k<k'} \varepsilon_k b_k^* b_{k'}^* \frac{\gamma}{2} = \delta_{kk'} , \quad \sum_k b_k b_k^* \frac{\gamma}{2} = 0.$$}

Rewrite the Hamiltonian:

$$H = \sum_k \varepsilon_k C_k^* C_k + \sum_{k<k'} \lambda_{kk'} C_k^* C_{k'} + \sum_{k<k'} \lambda_{kk'} C_k^* b_{k'}^*$$

$$+ \sum_{k<k'} \lambda_{kk'} b_{k'}^* C_{k'} + \sum_{k<k'} \lambda_{kk'} b_{k'}^* b_{k'}^*$$

where the sum over all electron states $k$ has been written as the sum over two sets of states.

$$\sum_{k<k'} \varepsilon_k b_k^* b_{k'}^* = -\sum_{k<k'} \varepsilon_k b_k^* b_{k'}^* + \sum_{k<k'} \varepsilon_k$$

The last term in the right is a zero-point energy which may be neglected by changing the zero of the energy.
The 4 terms bilinear in the creation and destruction operators yield the following processes in first order:

Convention: An arrow pointing in the direction of increasing time indicates an electron; an arrow pointing in the direction of decreasing time indicates a hole.

2nd order contributions

Consider the following. We have 2 diagrams that contribute in 2nd order:

Now to conserve energy, in the initial state there is a phonon of energy \( w_1 \), and in the final state there is a phonon of energy \( w_2 \), where

\[
E_1 + w_1 = E_2 + w_2
\]

\( E_1 \) = energy of electron 1, \( E_2 \) = energy of electron 2.
From the perturbation theory of last term, the 2\textsuperscript{nd} order contribution to the amplitude is

\[
\sum_r \frac{\langle f | V | r \rangle \langle r | V^\dagger | i \rangle}{E_r - E_i + i\varepsilon},
\]

where the sum runs over all intermediate states r.

Evaluating (A) and (B):

\begin{align*}
(A): \quad & \sum_k \frac{\langle 21 | \lambda_{2k} C_{2k}^+ C_{k}^* | k \rangle \langle k | \lambda_k C_{k}^* C_{1}^+ | 1 \rangle}{E_1 - E_k + i\varepsilon} \\
& = \sum_k \frac{\lambda_{2k} \lambda_k}{E_1 + \omega_1 - E_k + i\varepsilon}
\end{align*}

\begin{align*}
(B): \quad & \sum_k \frac{\langle 21 | \lambda_{k} b_{k} C_{1} | 1 \rangle \langle 1 | b_{k}^\dagger C_{1}^+ | 2 \rangle \langle 2 | \lambda_{2k} C_{2k}^+ b_{2k}^\dagger | 1 \rangle}{E_1 + \omega_1 - (E_1 + \omega_1 + \omega_2 - E_k) + i\varepsilon} \\
& = \sum_k \frac{\langle 21 | \lambda_{k} b_{k} C_{1} | 1 \rangle \langle 1 | b_{k}^\dagger C_{1}^+ | 2 \rangle \langle 2 | \lambda_{2k} C_{2k}^+ b_{2k}^\dagger | 1 \rangle}{E_1 + \omega_1 - E_k + i\varepsilon} \\
& = -\sum_k \frac{\lambda_{2k} \lambda_k}{E_k - \omega_1 - E_1 + i\varepsilon} = \sum_k \frac{\lambda_{2k} \lambda_k}{E_1 + \omega_1 - E_k - i\varepsilon}
\end{align*}

So that the contribution of (B) is the same as that of (A) except for the sign of i\varepsilon in the denominator.

Notice that we have not evaluated the complete 2\textsuperscript{nd} order amplitude as we have omitted
The contribution of the holes may be included as a modification of the propagator.

Let \( \Lambda_{k'\alpha} = \int \phi_{k'}^* (x) \Lambda (x) \phi_k (x) \, dx \)

then the contribution of (A) may be written

\[
\int dx_2 \, dx_1 \, dt_1 \, dt_2 \, \phi_{k_2}^* (x_2) \Lambda (x_2) K_{2,1} (x_2, x_1) \phi_k (x_1)
\]

and from this derive where

\[
K_{2,1} = \Theta (t_2 - t_1) \sum_{\alpha} e^{i \mathcal{E}_\alpha (t_2 - t_1)} \phi_{k_2}^* (x_2) \phi_k (x_1)
\]

where \( \Theta (t_2 - t_1) = \begin{cases} 1 & t_2 > t_1, \\ 0 & t_2 < t_1, \end{cases} \)

If we define for \( t_2 < t, \)

\[
K_{2,1} = -\sum_{\alpha} e^{-i \mathcal{E}_\alpha (t - t_2)} \phi_{k_2} (x_2) \phi_k^* (x_1)
\]

(A) - (B) may be written in the same form as (A) alone:

\[
\int dx_2 \, dx_1 \, dt_1 \, dt_2 \, \phi_{k_2}^* (x_2) \Lambda (x_2) K_{2,1} (x_2, x_1) \phi_k (x_1).
\]
Relativistic Quantum Mechanics

There is no known violation of the invariance principle of relativity. Two consequences of putting together relativity and quantum mechanics that are important in elementary particle theory are: (1) The existence of antiparticles. Moreover, the dynamics of antiparticles are determined from those of the dynamics of the corresponding particles. This is known as the CPT theorem.

(2) The connection between spin and statistics, namely that integer-spin particles obey Bose statistics and that half-integer spin particles obey Fermi statistics. We will give a demonstration of this result later.

The Schrödinger equation \( H \psi = i \frac{\partial \psi}{\partial t} \) is unsymmetrical in its treatment of time and space.

Basically there are 3 ways of doing relativistic quantum mechanics:

(1) A method which is easy to understand, but in which the relativity is not manifestly obvious.

(2) A method which is less easy to understand, in which the relativity is obvious.

(3) Abstract, elegant way.

We will follow method (1). In some problems, approach (2) is simpler.

Maxwell's Equations are relativistically invariant, but their effects may be treated in a non-relativistic manner in some applications (Compton scattering, photoelectric effect).

Maxwell's Eqs. may be derived from a Lagrangian \( L \) that gives the action

\[
\text{action} = \int \left[ (\nabla \phi)^2 - (\nabla A)^2 \right] d^3x \, dt
\]
Interaction with matter is handled by adding the additional term $\int j_\mu(x) A_\mu(x) \, dx$ to the Lagrangian.

In the noncovariant noncovariant Coulomb gauge $\nabla \cdot A = 0$, the interaction of charged particles through the electromagnetic field is equivalent to

1. Instantaneous Coulomb interaction $\frac{e^2}{r_{ij}}$.

2. $H = \sum_k \mathbf{A}_k \cdot \mathbf{A}_k + \int j_\mu(x) A_\mu(x) \, dx + H_{\text{matter}}$

$$A(x) = \sum_k \frac{1}{k_i} \left[ \varepsilon_{kl} \varepsilon_{ik} e^{ik \cdot x} + \varepsilon_{kl} \varepsilon_{ik} e^{-ik \cdot x} \right]$$

Let us write down the classical relativistic relations for particles:

$$E = m_0 \gamma \quad p = m_0 \gamma \mathbf{v} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

(c. taken to be 1)

From here on down, it becomes convenient to omit the subscript $\mu$, all indices referred to will be understood.

$$E^2 = p^2 + m_0^2$$

or

$$p_\mu = M$$

where $p_\mu = (E, \mathbf{p})$

and if $A_\mu = (A_0, \mathbf{A})$, $B_\mu = (B_0, \mathbf{B})$

Then $A \cdot B = A_\mu B_\mu = A_0 B_0 - \mathbf{A} \cdot \mathbf{B}$

Definition: $\delta_{\mu \nu} = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2, \text{ or } 3 \end{cases}$

Let $\nabla_\mu = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = (\frac{\partial}{\partial t}, -\mathbf{\nabla})$ (Notice the minus sign in the definition of $\nabla_\mu$)

Another useful vector is $\mathbf{x}_\mu = (t, x, y, z)$
Non-relativistically the phase of a free particle of momentum \( k \) frequency (energy) \( \omega \) is
\[
e^{i(k \cdot x - \omega t)} \quad \text{or} \quad e^{i(k \cdot x - \omega t)}
\]
This may be written as a Lorentz invariant as
\[
e^{-ip \cdot x}
\]
we will take this over to the relativistic case (first done by de Broglie).

Then, using the relativistic expression:
\[
is = \sqrt{k^2 + m^2}
\]
we obtain
\[
\frac{d\psi}{dk} = \frac{d\psi}{dp} = \frac{dE}{dp} = \frac{E}{c} = \frac{P}{E}
\]

The Schrödinger \( \dot{\psi} \) is of first order in time and permits the principle of superposition which we want to retain. Let us look for a relativistic wave equation that is first order in time.
\[
E = \sqrt{p^2 + m^2}
\]
suggests
\[
\frac{i\psi}{\sqrt{m^2 - c^2}} = \frac{\psi}{c} \quad \text{which, horrible as it looks, can be made to work out.}
\]

But the equation in the presence of an external field:
\[
\left( i \frac{\partial}{\partial t} - eA_z \right) \psi = \sqrt{m^2 - (E - icA_z)^2} \psi
\]
presents difficulties.

A second-order equation suggests itself:
\[
\left[ - \left( i \frac{\partial}{\partial t} - eA_z \right)^2 - \frac{E^2}{c^2} \right] \psi = m^2 \psi
\]
This is a serious contender for a relativistic wave equation. There are possible difficulties:

1. Does it conserve probability?
2. Does it reduce to a correct non-relativistic equation? (in the non-relativistic limit)
3. Energy levels of the hydrogen atom.
4. Does it violate causality?

Problem (due first week next term): Test out one of these questions.

The most serious test is 3 — the others are basically theoretical prejudices.

The Dirac Equation

Dirac looked for an equation of the form \( H \psi = i \frac{\partial \psi}{\partial t} \), where he permitted the wave-function \( \psi \) to have several components, and the Hamiltonian was permitted to be a matrix operator on these components.

\[ H = \alpha_x \frac{1}{i} \frac{\partial}{\partial x} + \alpha_y \frac{1}{i} \frac{\partial}{\partial y} + \alpha_z \frac{1}{i} \frac{\partial}{\partial z} + \beta m \]

where \( \alpha, \beta \) are matrices.

i.e., \( H = \alpha \cdot \mathbf{p} + \beta m \quad \alpha, \beta \) hermitean

an external
with an electric field present,

\[ H = \alpha \cdot (\mathbf{p} - e \mathbf{A}) + \beta m \quad p = -i \frac{\partial}{\partial t} \]

and \( \frac{id}{dt} \rightarrow i \frac{\partial}{\partial t} - e \phi \)
Look for a sol'n of the form \( \psi = u e^{-ik \cdot x} \) (free particle).

Then \( u \) satisfies the equation

\[
E u = (\mathbf{a} \cdot \mathbf{k} + \beta m) u.
\]

\[
\Rightarrow d_x^2 + d_y^2 + d_z^2 = 1 = d_y^2 = d_z^2
\]

\[
d_x d_y + d_y d_x = 0, \text{ etc.}
\]

\[
\beta d_x + d_x \beta = 0
\]

and \( \beta = 1 \)

In 4 dimensions an explicit set of \( \mathbf{a} \)'s and \( \beta \) that satisfy these equations is

\[
\mathbf{a}_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6_x \\ 6_x & 0 \end{pmatrix}
\]

\[
\mathbf{a}_y = \begin{pmatrix} 0 & 6_y \\ 6_y & 0 \end{pmatrix}, \quad \mathbf{a}_z = \begin{pmatrix} 0 & 6_z \\ 6_z & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]

Where \( 6_x, 6_y, 6_z \) are the usual Pauli spin matrices. \( I \) is 2x2 identity matrix. \( 0 \) is 2x2 zero matrix.

Write the 4 component wavefunction

\[
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} e^{-ik \cdot x} \quad \text{as} \quad \begin{pmatrix} u_a \\ u_b \end{pmatrix} e^{-ik \cdot x}
\]

where \( u_a, u_b \) are 2 component objects.
In an external field \( (i \frac{\partial}{\partial t} - e\phi) \psi = \alpha \left( \frac{i}{2} \nabla - eA \right) \psi + \beta m \psi \)

"multiply" by \( \beta \) : \( \beta (i \frac{\partial}{\partial t} - e\phi) + \beta \alpha \left( i \nabla + eA \right) \psi = m \psi \)

Define another set of matrices: \( \chi_0 = \gamma_t = \beta \)
\( \chi = \beta \gamma \)

Then the equation becomes
\[ \chi_\mu \left( i \nabla_\mu - eA_\mu \right) \psi = m \psi \]

where \( \chi_\mu \chi_\nu + \chi_\nu \chi_\mu = 2 \delta_\mu^\nu \)

These commutation rules are invariant under parity transformations.

Let \( \psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \)

then \( \left( -\frac{1}{i} \frac{\partial}{\partial t} - e\phi \right) \psi_a = \alpha \left( \frac{i}{2} \nabla - eA \right) \psi_b + m \psi_a \)

and \( \left( -\frac{1}{i} \frac{\partial}{\partial t} - e\phi \right) \psi_b = \alpha \left( \frac{i}{2} \nabla - eA \right) \psi_a - m \psi_b \)

For a neutrino, the equations are
\[ -\frac{1}{i} \frac{\partial}{\partial t} \psi_a = \alpha \left( \frac{i}{2} \nabla \right) \psi_b \]
\[ -\frac{1}{i} \frac{\partial}{\partial t} \psi_b = \alpha \left( \frac{i}{2} \nabla \right) \psi_a \]
DIRAC EQUATION

\[-\gamma_0 \frac{\partial \psi}{\partial t} = \mathbf{H} \psi\]

where

\[\mathbf{H} = \beta m + e \mathbf{V} + \mathbf{A} \cdot (\mathbf{p} - e \mathbf{A})\]

in order to a) have an equation linear in time derivatives and b) have time and space derivatives enter on an equal footing.

It is found that in order for this to work, different \(\gamma_i\) must anticommute, and \(\gamma_i \gamma_j + \beta \gamma_j \gamma_i = \delta_{ij}\). One representation of these commutation relations is

\[\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\]

Hence if \(\psi\) is written in the form of a pair of 2 component spinors \(\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}\), then the Dirac equation gives a set of 2 coupled equations for the two component spinors.

Because \(\overline{\psi} = \psi^\dagger\psi\) is Hermitian, \(\frac{\partial}{\partial t} \int \overline{\psi} \psi \, d\mathbf{v} = \int \left[ \frac{\partial}{\partial t} \overline{\psi} \psi + \psi^\dagger \frac{\partial}{\partial t} \psi \right] \, d\mathbf{v} = \int \left[ -\psi^\dagger \frac{\partial}{\partial t} \psi + \psi^\dagger \frac{\partial}{\partial t} \psi \right] \, d\mathbf{v} = 0\)

Thus \(\overline{\psi} \psi\) can be thought of as a probability density.

Expectation values take the usual form \(\langle \psi \rangle = \int \psi^\dagger \chi \psi \, d\mathbf{v}\).

Notice that none of the answers will change if we change the representation (i.e., if we pick a different form which satisfies the commutation relations) for the \(\sigma\) and \(\beta\) matrices. Then the new representation can be related to the old as follows:

\[S \psi = \psi' \quad \sigma^0 = \sigma^0 \quad \gamma^0 = -\gamma^0\]

\[\beta' = \beta S^* \quad \sigma^i = S \sigma^i S^* \]

Check:

\[\overline{\psi'} \psi' = \int \overline{\psi'} \psi' \, d\mathbf{v} = \int \overline{\psi} \psi \, d\mathbf{v} = \langle \psi \rangle\]

Change of representation sometimes makes the physics more transparent. For example, the Dirac equation written in terms of \(\psi_a\) and \(\psi_b\) looks like

\[\left( E - eV \right) \psi_a - \overrightarrow{\mathbf{\sigma}} \cdot \overrightarrow{\mathbf{p}} \psi_b = m \psi_a \]

\[\left( E - eV \right) \psi_b - \overrightarrow{\mathbf{\sigma}} \cdot \overrightarrow{\mathbf{p}} \psi_a = -m \psi_b \]

for \(\psi_s = \psi_a + \psi_b\) \quad \psi_d = \psi_a - \psi_b\)

these may be rewritten as

\[\left( E - eV \right) \psi_s - \overrightarrow{\mathbf{\sigma}} \cdot \overrightarrow{\mathbf{p}} \psi_a = m \psi_s \]

\[\left( E - eV \right) \psi_d + \overrightarrow{\mathbf{\sigma}} \cdot \overrightarrow{\mathbf{p}} \psi_a = m \psi_d \]
\[
(E - eV - \vec{\sigma} \cdot \vec{p}) (E - eV - \vec{\sigma} \cdot \vec{p}) \psi_5 = m^2 \psi_5 \\
(E - eV - \vec{\sigma} \cdot \vec{p}) (E - eV + \vec{\sigma} \cdot \vec{p}) \psi_4 = m^2 \psi_4
\]

Thus in this representation it is easy to uncouple the equations (although the equation has become quadratic in the energy rather than linear, in the process).

Because the electron in \( \beta \) decay is created as pure \( \psi_5 \), this is the natural representation to use to calculate the effects of electromagnetism on it as it shoots out of the nucleus.

**Change of Notation**

To make the equation more obviously similar in time and space components, multiply through by \( \beta \)

\[
\left\{ \beta \left[ -\frac{\sigma \cdot \vec{p}}{2} - eV \right] - \beta \vec{\sigma} \cdot (\vec{p} - c \vec{A}) \right\} \psi = m \psi
\]

Define \( \psi_0 = \beta \)

\[
\vec{\beta} = \beta \vec{\sigma} = \left( \begin{array}{cc} \frac{\sigma^y}{\sigma^0} \end{array} \right)
\]

because \( \rho_0 = -\frac{\sigma \cdot \vec{p}}{2} \)

we can write

\[
\left[ \psi_0 (\rho_0 \cdot \vec{p} - eV) - \vec{\sigma} \cdot (\vec{p} - c \vec{A}) \right] \psi = m \psi
\]

For any four vector \( \vec{B} \), define \( \vec{B} = B_0 \vec{\sigma} - \vec{B} \cdot \vec{\sigma} \).

then the Dirac equation takes the form

\[
\left[ \left( \vec{\sigma} c - \vec{\varepsilon} \vec{A} - m \right) \psi = 0 \right]
\]

i.e. because \( \rho_0 = i \frac{\sigma \cdot \vec{p}}{2} \) \( \vec{p} = \frac{\sigma \cdot \vec{p}}{2} \) define \( \nabla \mu = \left( \frac{\partial}{\partial \tau}, -\frac{\partial}{\partial x} \right) \)

then

\[
(i \nabla c - \vec{\varepsilon} \vec{A}) \psi = m \psi
\]

**Solutions of the Dirac Equation**

a) Particle at rest, no field

\( (\gamma_0 \psi_0 - m) \psi = 0 \Rightarrow \left( \begin{array}{cc} E-m & 0 \\ 0 & -(E+m) \end{array} \right) \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = 0 \)

For \( \psi_+ \), \( E=m \)

\[
\psi_+ = \left( \begin{array}{c} \phi \\ 0 \end{array} \right) e^{-im \tau}
\]

For \( \psi_-, E=-m \)

\[
\psi_- = \left( \begin{array}{c} 0 \\ \chi \end{array} \right) e^{im \tau}
\]

Here \( \phi \) and \( \chi \) can be any 2 component spinor \( \left( \begin{array}{c} \phi \\ \chi \end{array} \right) \) for general constants a and b)

b) No field, particle moving

It is much easier to solve the problem in momentum space. Then one can always Fourier transform back by multiplying by \( e^{-iE t + i \vec{p} \cdot \vec{x}} \).

\[
(\vec{\sigma} c - m) \psi = 0
\]
\[(E - m)\phi - \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \chi = 0\]
\[\frac{\vec{p} \cdot \vec{\sigma} \phi - (E + m) \chi = 0}{E + m}\]

**Solution i)** \[\chi = \left(\frac{\vec{p} \cdot \vec{\sigma}}{E + m}\right) \phi\]

\[\chi = \begin{pmatrix} \phi \\ \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi \end{pmatrix}\]

By substituting this back we see that
\[(E - m)\phi - \frac{\vec{p} \cdot \vec{\sigma} \phi - (E + m) \chi}{E + m} = 0\]
\[E^2 = p^2 + m^2\]

Hence in position space
\[\chi = \begin{pmatrix} \phi \\ \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi \end{pmatrix} e^{i(t + \vec{p} \cdot \vec{x})} + i\vec{p} \cdot \vec{x}\]

Choose the sign such that as \(\vec{p} \rightarrow 0\), this approaches
\[\chi = \begin{pmatrix} \phi \\ \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi \end{pmatrix} e^{-iE t} e^{-i\vec{p} \cdot \vec{x}}\]

**Solution ii)** \[\phi = \frac{\vec{p} \cdot \vec{\sigma} \chi}{E - m}\]

Here again \(s^2 = m^2 + p^2\), but we choose the square root such that as \(\vec{p} \rightarrow 0\), the solution approaches
\[\begin{pmatrix} 0 \\ \chi \end{pmatrix} e^{-iE t}\]

Thus we get
\[\begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma} \chi}{m + \sqrt{p^2 + m^2}} \\ \chi \end{pmatrix} e^{i\sqrt{p^2 + m^2} t + i\vec{p} \cdot \vec{x}}\]

or, if \(\vec{p}' = -\vec{p}\),
\[\begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma} \chi}{m + \sqrt{p^2 + m^2}} \\ \chi \end{pmatrix} e^{i\sqrt{p^2 + m^2} t - i\vec{p} \cdot \vec{x}}\]

which is sometimes written
\[\begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma} \chi}{m + \sqrt{p^2 + m^2}} \\ \chi \end{pmatrix} e^{-iEt + i\vec{p}' \cdot \vec{x}}\]

Thus if we associate this second solution with \(\vec{p}'\), it looks just like the first solution except that the "large" and "small" components of the spinor have been interchanged, and there is an overall - sign in the exponent. Some people like to think of this - sign as a reversal of \(x\) and \(t\); this at least is a mnemonic.

For the moment, ignore the problem of normalizing these.

It is much more complicated to find solutions for \(\vec{F} \neq 0\). For the present, let's forget about this and just try to understand the nonrelativistic limit of the Dirac equation in a field.
\[ \left( H - E \right) \psi = 0 \]

reduces to the set of coupled equations
\[ (E - m - \varepsilon_v \phi - (E^2 - \varepsilon_v \phi) \cdot \hat{\omega} \phi = 0 \]
\[ \hat{\omega} \cdot (\hat{p} - \varepsilon_v \phi) \cdot \hat{\omega} \phi = (E + m - \varepsilon_v \phi) \phi \]
\[ \chi = \left( \frac{E + m - \varepsilon_v \phi}{E + m - \varepsilon_v \phi} \right) \hat{\omega} \cdot (\hat{p} - \varepsilon_v \phi) \phi \]

Substitute this into the first equation. It is important to preserve the order of operators, because the \( \hat{p} \) and \( E \) are differential operators which don't commute with \( \phi \).

Use \( E \equiv M + W \)
\[ \frac{1}{2M + W - \varepsilon_v \phi} \approx \frac{1}{2M} - \frac{(W - \varepsilon_v \phi)}{(2M)^2} \]

\[ (W - \varepsilon_v \phi) \phi = \frac{1}{2M} \hat{\omega} \cdot (\hat{p} - \varepsilon_v \phi) \hat{\omega} \phi - \frac{1}{2M} (W - \varepsilon_v \phi) \hat{\omega} \cdot \hat{\omega} \phi \]

Rewrite the equation by adding and subtracting the same thing on both sides
\[ \left[ 1 + \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right] (W - \varepsilon_v \phi) \left[ 1 + \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right] \phi = \frac{1}{2M} \left( \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right) \phi + \]
\[ + \frac{1}{2M} \left[ (\hat{\omega}, \hat{\pi})^2 \left( W - \varepsilon_v \phi \right) - \frac{1}{2M} \left( W - \varepsilon_v \phi \right) (\hat{\omega}, \hat{\pi})^2 + (W - \varepsilon_v \phi) \left( \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right) \phi \right. \]

To really be able to consider the non-relativistic limit, we should write everything in terms of a non-relativistic wave function. \( \phi \) is not the non-relativistic wave function in the case where the fields are present. Define \( \mathcal{A} = \left[ 1 + \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right] \phi \)

Then to lowest non-trivial order \( \phi = \mathcal{A} - \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \mathcal{A} \)

hence \( \mathcal{A} \mathcal{A} = \phi^T \phi + \mathcal{A} \mathcal{A} \approx \phi^T \phi \left[ 1 + \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right] \approx \mathcal{A}^T \mathcal{A} \)

\( \mathcal{A} \) contains the major effects of the "small" components as well as the large ones. Thus we can think of \( \mathcal{A} \) as a non-relativistic wave function and imagine the problem of finding the energy eigenvalues to be the same as that of minimizing
\[ \frac{\mathcal{A}^T \text{ (mess) } \mathcal{A}}{\mathcal{A}^T \mathcal{A} \text{ dual}} \]

Upon substituting this expression for \( \phi \) we obtain an equation of the form
\[ \left[ 1 + \frac{(\hat{\omega}, \hat{\pi})^2}{2M^2} \right] (W - \varepsilon_v) \mathcal{A} = (\text{ something}) \]

which we unravel again to first order by
\[ (W - \varepsilon_v) \mathcal{A} \approx \text{ something} - \frac{(\hat{\omega}, \hat{\pi})^2}{2M^3} \mathcal{A} \]

Keeping lowest order terms, we arrive at
\[ (W - \varepsilon_v) \mathcal{A} = \frac{1}{2M} \left( \frac{(\hat{\omega}, \hat{\pi})^2}{2M^3} \mathcal{A} - \frac{(\hat{\omega}, \hat{\pi})^2}{2M^3} \mathcal{A} \right) \]
\[ + \frac{1}{2M} \left\{ (\hat{\omega}, \hat{\pi})^2 (W - \varepsilon_v) \mathcal{A} - \frac{(\hat{\omega}, \hat{\pi})^2}{2M^3} (W - \varepsilon_v) \mathcal{A} + (W - \varepsilon_v) \left( \frac{(\hat{\omega}, \hat{\pi})^2}{2M^3} \right) \mathcal{A} \right\} \]
For any two operators \( A \) and \( B \),
\[
\]
set \( A = \frac{\sigma \cdot \mathbf{p}}{m} \)
\( B = W - V \)
Calculate
\[
[ \frac{\sigma \cdot \mathbf{p}}{m}, E - m - eV ] = \left[ \sigma \cdot ( \mathbf{p} - e \mathbf{A} ), E - m - eV \right]
\]
Assume for simplicity that \( \frac{e}{m} \) (This is not necessary. For fun and practice try working the whole mess through without assuming this)
Thus we get
\[
- e \left[ \frac{\partial}{\partial \mathbf{r}}, \mathbf{p} \cdot \mathbf{E} \right] = - e \sigma \cdot [ \mathbf{p}, \mathbf{E} ] = - i e \sigma \cdot \mathbf{E}
\]
\[
\{ \} = [ \frac{\sigma \cdot \mathbf{p}}{m}, - e \sigma \cdot \mathbf{E} ]
\]
\[
- i e \left[ \frac{\partial}{\partial \mathbf{r}}, \mathbf{p} \cdot \mathbf{E} \right] + \frac{e^2}{2} \left( \frac{\sigma \cdot \mathbf{A}}{m} \frac{\sigma \cdot \mathbf{E}}{m} - \frac{\sigma \cdot \mathbf{E} \cdot \mathbf{A}}{m^2} \right)
\]
use
\[
\frac{\sigma \cdot \mathbf{A}}{m} \frac{\sigma \cdot \mathbf{E}}{m} = \frac{\mathbf{A}}{m} + \frac{i \sigma \cdot ( \mathbf{A} \times \mathbf{E} )}{m^2}
\]
\[
\left[ \frac{\sigma \cdot \mathbf{A}}{m} \frac{\sigma \cdot \mathbf{E}}{m} \right] = 2 i \frac{\sigma \cdot ( \mathbf{A} \times \mathbf{E} )}{m^2}
\]
\[
\left[ \frac{\sigma \cdot \mathbf{p}}{m} \frac{\sigma \cdot \mathbf{E}}{m} \right] = - i \nabla \cdot \mathbf{E} + i \frac{e}{m} \left( \mathbf{p} \times \mathbf{E} \right) - i \sigma \cdot ( \mathbf{E} \times \mathbf{p} )
\]
\[
\frac{\sigma \cdot ( \mathbf{p} \times \mathbf{E} )}{m^2} f = \frac{e}{m} \frac{\partial}{\partial \mathbf{r}} \mathbf{E} \cdot f + \frac{e}{i m} \left[ \frac{\partial \mathbf{E}}{\partial \mathbf{r}} - \frac{\partial \mathbf{E}}{\partial \mathbf{r}} \right] f
\]
But
\[
\mathbf{E} \mathbf{f} = - \frac{\partial \mathbf{f}}{\partial \mathbf{r}} + \nabla \times \mathbf{E} \mathbf{f} = - \nabla \times \mathbf{E} \mathbf{f}
\]
and
\[
- i e \left[ \frac{\partial}{\partial \mathbf{r}}, \mathbf{p} \cdot \mathbf{E} \right] = - e \left[ \nabla \cdot \mathbf{E} + 2 \frac{\sigma \cdot ( \mathbf{E} \times \mathbf{p} )}{m^2} \right]
\]
In similar fashion we calculate
\[
\frac{\sigma \cdot \mathbf{p}}{m} \frac{\sigma \cdot \mathbf{p}}{m} = \frac{\mathbf{p} \cdot \mathbf{p}}{m^2} + i \frac{\sigma \cdot ( \mathbf{p} \times \mathbf{p} )}{m^2}
\]
\[
\frac{\mathbf{p} \times \mathbf{p}}{m^2} = \nabla \times \mathbf{A} = \nabla \times \nabla \mathbf{A} = \nabla \times \nabla \left[ \frac{1}{2} \nabla \cdot \mathbf{p} \mathbf{p} \right]
\]
Hence we obtain (to lowest order)
\[
\begin{align*}
\mathbf{w} \mathbf{A} &= \mathbf{V} \mathbf{A} + \frac{1}{2 m} ( \nabla \cdot \mathbf{p} \mathbf{p} ) \mathbf{A} - \frac{c^2}{2 m} \frac{\sigma \cdot \mathbf{B}}{m^2} \mathbf{A} - \frac{( \mathbf{p} \cdot \mathbf{p} )}{8 m^3} \mathbf{A} \\
- \frac{e}{8 m^2} \left( \nabla \cdot \mathbf{E} + 2 \frac{\sigma \cdot ( \mathbf{E} \times \mathbf{p} )}{m^2} \right) \mathbf{A}
\end{align*}
\]
The individual terms can now be interpreted
\[
\begin{align*}
\mathbf{V} &= \text{ordinary potential energy} \\
\frac{\mathbf{p} \cdot \mathbf{p}}{2 m} &= \text{non-relativistic kinetic energy} \\
- \frac{c^2}{2 m} \frac{\sigma \cdot \mathbf{B}}{m^2} &= \text{Pauli spin effect (magnetic moment \( \frac{e c^2}{2 m} \) due to spin)} \\
- \frac{( \mathbf{p} \cdot \mathbf{p} )}{8 m^3} &= \text{first relativistic correction to the kinetic energy}
\end{align*}
\]
\[
\left( \sqrt{\mathbf{p}^2 + m^2} = m \sqrt{1 + \frac{\mathbf{p}^2}{m^2}} = m \left[ 1 + \frac{1}{2} \frac{\mathbf{p}^2}{m^2} - \frac{1}{8} \frac{\mathbf{p}^2}{m^4} + \cdots \right] \right)
\]
We are left only with the term

\[
\frac{-e}{2M} \left\{ \frac{\nabla \cdot \vec{E}}{4\pi} + \frac{\vec{B} \cdot (\vec{E} \times \vec{n})}{2M} \right\}
\]

is like \( \frac{\vec{p} - q\vec{A}}{m} \) a particle in motion in an electric field sees a magnetic field

\[-\frac{\vec{E} \times \vec{n}}{c} = \vec{E} \times \vec{B}/m \]

Then the energy of interaction of the electron's spin magnetic moment with this field will be

\[-\frac{e}{2M} \frac{\vec{B} \cdot (\vec{E} \times \vec{n})}{m} \]

This looks like the term found above but differs from it by a factor of 2 (see below).

(Another way to get the same result is to recall that a moving magnetic moment produces an electric dipole moment of the form \(-\mu \times \vec{v}\) which then interacts with the electric field via \(-\mu \times \vec{E}\)).

Within an atom, the electric field of the nucleus has the form \(\frac{\vec{r}}{r^3}\).

Then \(\vec{\mathcal{L}} \cdot (\vec{r} \times \vec{E}) \propto \frac{2e^2}{4\pi M^2} \frac{(\vec{r} \times \vec{p}) \cdot \vec{B}}{r^3}\)

\(\vec{p} \times \vec{B} = \vec{L}\) the orbital angular momentum

\(\vec{\mathcal{L}} \cdot \vec{\omega} \propto \frac{1}{r^3}\)

produces \(\vec{\mathcal{L}} \cdot \vec{\omega} \propto \frac{1}{r^3}\)

\(\vec{\omega} \times \vec{S} = \mathcal{J} (\mathcal{J} + 1) - \ell (\ell + 1) - s (s + 1)\) splits apart energy levels of different \(\ell\).

\(\vec{\mathcal{L}} \cdot \vec{\omega}\) is, however, 0 for \(L=0\) states.

But for \(\vec{E} = \frac{Ze^2}{r^3}\), \(\nabla \cdot \vec{E} = 4\pi \frac{Ze}{r} \)

This contributes only to \(s\) states (the only ones which are non-zero at the origin).

Thus the combination of terms in \(\{\}\) can be thought of as a spin-orbit contribution, for every \(l\).

Not all spin \(\frac{1}{2}\) particles found in nature have magnetic moments equal to the Dirac moment \(\frac{g}{2m}\). In fact only the electron and the muon appear to obey the unmodified Dirac equation (in any approximation). To account for these "anomalous" magnetic moments, an additional term is added to the original equation (Pauli's idea).

Of course, you could add lots of different terms to the original equation and still have it remain Lorentz covariant, but it is conventional to choose a particular \(A\) form which accounts for the observations (and introduces as few derivatives as possible. This second criterion will become reasonable in a month or two when we study various divergences.)
The new equation is
\[ \left[ \mathfrak{f} - e A - \frac{\hbar}{2} \varepsilon_{\mu \nu} F^{\mu \nu} - m \right] \psi = 0 \]
where
\[ \varepsilon_{\mu \nu} = \frac{i}{2} \left[ \gamma_{\mu}, \gamma_{\nu} \right] \]
\[ F_{\mu \nu} = \frac{\partial A_{\nu}}{\partial x_{\mu}} - \frac{\partial A_{\mu}}{\partial x_{\nu}} \]

Figure out how to work this through to a non-relativistic equation and show that (up to a possible sign)
\[ \nu = V' \mathfrak{f} + \frac{1}{2m} \left( \frac{\hbar^2}{\mu} \right) \mathfrak{f} + \left( \mu - \frac{e^2}{2m} \right) \frac{\hbar^2}{\mu} \mathfrak{f} - \frac{(\mathbf{p} \cdot \mathbf{A})^2}{\mu} \]
\[ + \frac{1}{2} \left( \mu - \frac{e^2}{4m} \right) \left( \frac{\hbar^2}{\mu} \mathfrak{f} + 2 \mathbf{p} \cdot (\mathbf{E} \times \mathbf{n}) \right) \]

This form shows that the factor of 2 which we previously had trouble with in the term is not really an overall multiplier. Rather there are two effects
\[ \frac{1}{2} \left( \mu - \frac{e^2}{4m} \right) \left\{ \mathfrak{f} \right\} + \frac{1}{2} \frac{e^2}{4m} \left\{ \mathfrak{f} \right\} \]
This part was explained above.

The second term is called the Thomas term. Its presence may be understood as follows:

We calculated the effect on the electron due to precession of the spin magnetic moment in the magnetic field seen instantaneously. This would be ok if the electron were an inertial frame. But it is constantly being accelerated by a force proportional to the charge. Thus we must correct for this (since the acceleration is not proportional to magnetic moment, we don't expect the correction term to involve \( \mu \)).

The correction is actually calculated in the Ph 209 book. You might be able to find a dimpler and clearer explanation of the numbers on your own.

When the magnetic moments of the electron and muon are actually measured, they are found to be slightly different from \( \frac{e}{2m} \). The discrepancy can, however, be completely accounted for by considering quantum effects of the electromagnetic field.

The anomalous magnetic moments of other spin \( \frac{1}{2} \) particles are presumably due to their interactions with mesons. One way of looking at the situation is to say that the physical proton (neutron, lambda, etc.) is made up of an ideal Dirac proton plus a lot of mesons, which are continually being emitted and absorbed. These mesonic interactions alter the Dirac moment to its anomalous value. At present there is no good (i.e. precise) way to theoretically calculate anomalous moments.
Anomalous magnetic moment of the electron:

An electron is described by a wave-function $\psi$ that satisfies the Dirac Equation:

$$\gamma_\mu \left( i \partial_\mu - e A_\mu \right) \psi = m \psi \quad \cdots \cdots \quad (1)$$

Where $A_\mu = A^\text{total}_\mu$ is made up of an external field $A^\text{ext}_\mu$ together with $A^\prime_\mu$, the field due to the electron itself.

$$A_\mu = A^\text{ext}_\mu + A^\prime_\mu$$

Classically, an electron trajectory satisfies

$$m \frac{d^2 x_\nu}{d\tau^2} = \gamma_\mu F^\nu_\mu$$

where $F^\nu_\mu = F^{\text{ext}}_\nu F^\nu_\mu$

then

$$m \frac{d^2 x_\nu}{d\tau^2} - \gamma_\mu F^\nu_\mu = \gamma_\mu F^{\text{ext}}_\nu$$

and the term $\gamma_\mu F^\prime_\mu$ is called the force of radiation resistance.

Equation (1) with $A_\mu = A^\text{ext}_\mu$ is analogous to the classical equation in which the force of radiation resistance is neglected.

If we put the full $A_\mu$ instead of $A^\text{ext}_\mu$ into eq. (1) we obtain the anomalous magnetic moment of the electron as a term in the non-relativistic reduction of the hamiltonian derived from eq. (1). The term would look like

$$\frac{e}{2m} g \cdot \vec{E}^\text{ext}$$

where the anomalous moment $g$ is determined from the radiation resistance. The calculation of this quantity using standard techniques yields divergences which may be avoided by the method of “mass and charge renormalization.”
One compares the calculated value of the anomalous moment with experiment by observing the hyperfine structure of the energy levels of atomic hydrogen. One also wants to correct for effects due to a space charge distribution within the proton when making the comparison.

There is no experiment today that absolutely requires a modification of the scheme outlined above.

Problems

1. In the non-relativistic reduction of the Dirac equation with the Pauli term there appeared the term

\[ \frac{e}{8m} \text{ } \vec{V} \cdot \vec{E} \]

which comes mathematically from \( [\vec{e} \vec{p}, \vec{e} \vec{E}] \).

Explain the physical origin of the term.

2. For entertainment, you might try to examine the Dirac equation in Hamiltonian form with

\[ H_D = \vec{\alpha} \cdot (\vec{p} - e \vec{A}) + \beta \mu \]

and using the operator equation of motion

\[ \dot{\vec{\phi}} = i [H, \vec{\phi}] \]

evaluate \( \vec{\dot{\phi}} \) and \( \dot{\vec{\phi}} \).

Find the velocity operator for the Dirac electron.

3. How many different metrics can you make from products of \( \vec{\alpha} \)'s and \( \beta \)'s? (The purpose of this exercise is to gain familiarity with the commutation laws).
Solutions to the Dirac Equation for a Free Particle

Let \( \Psi = e^{-i(Et - \mathbf{p} \cdot \mathbf{x})} \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} \); \( \Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} \)

Then with our choice of \( \alpha = \begin{pmatrix} 0 & \sigma \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \end{pmatrix} \) the Dirac Equation becomes

\[
\begin{align*}
(E + m) \Psi_b &= \sigma \cdot \mathbf{p} \Psi_a \\
(E - m) \Psi_a &= \sigma \cdot \mathbf{p} \Psi_b
\end{align*}
\]

\[
(E^2 - m^2) \Psi_a = (E+m) \sigma \cdot \mathbf{p} \Psi_b = \sigma \cdot \mathbf{p} (E+m) \Psi_b = \mathbf{p}^2 \Psi_a
\]

so that \( E^2 = m^2 + \mathbf{p}^2 \) or \( \Psi = 0 \).

\[\Rightarrow E = \pm E_p \quad \text{where} \quad E_p = +\sqrt{\mathbf{p}^2 + m^2} \]

A. \( \frac{E = E_p}{\bar{E} = E_p} \)

\[\Psi_b = \frac{\sigma \cdot \mathbf{p}}{m + E_p} \Psi_a \]

Let us choose our axes so that \( \mathbf{p} \) is in \( \hat{z} \) direction.

Spin up (positive helicity)

then \( \mathbf{p} \cdot \Psi_a = + \Psi_a = \alpha(1) \quad \delta = \text{some number chosen for convenient normalization} \)

then \( \Psi_b = \frac{\sigma \cdot \mathbf{p}}{m + E_p} \Psi_a = \frac{\sigma \cdot \mathbf{p}}{E_p + m} \Psi_a \) \( \alpha(0) \)

Choosing the normalization \( \Psi^\dagger \Psi = \Psi_b^\dagger \Psi_b + \Psi_a^\dagger \Psi_a = 1 \)

we have \( \delta = \sqrt{E + m} \quad \text{or} \quad \delta = \sqrt{\frac{E + m}{2E}} \)

or \( \Psi = \sqrt{\frac{E + m}{2E}} \begin{pmatrix} 0 \\ \frac{\mathbf{p}}{E_p + m} \end{pmatrix} = \frac{1}{\sqrt{2E_p}} \begin{pmatrix} \sqrt{E_p + m} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{E_p - m}{E_p + m} \end{pmatrix} \)
Spin down  \( \hat{y}_a = -\hat{y}_a \)
(negative helicity)

then  \( \hat{y}_b = -\hat{y}_b \)

and  \[
\Psi = \frac{1}{\sqrt{2E_F}} \begin{pmatrix}
0 \\
\frac{1}{V E_F + m} \\
0 \\
-\frac{1}{V E_F + m}
\end{pmatrix}
\]

Now, if we want an eigenstate of \( \hat{g}_x \) instead of \( \hat{g}_y \), we can try  \( \hat{g}_x \hat{y}_a = \hat{y}_a \hat{g}_x = \frac{i}{\hbar} \hat{g}_x \)

but  \[
\Psi_b = \frac{\hat{p} \Psi}{\hat{p} (\hat{g}_x + m)} \]

is not an eigenstate of \( \hat{g}_x \).

This is because the operator \( (\hat{g}_y \hat{0} \hat{g}_x) \) does not commute with the hamiltonian. Only the operator \( (\hat{g}_y \hat{0} \hat{g}_y) \) commutes with \( H \) when the momentum is in the \( z \) direction.

The form of the 4x4 matrices that represent the spin operators are:

\[
(0 \ g \ 0) \quad \text{with our choice of } x, y, \text{ and}
\]

more generally,

\[
\hat{g}_x = -i \partial_y \hat{g}_y = -i \partial_y \hat{g}_x = -i \partial_x \hat{g}_y
\]

Momentum in arbitrary direction

Suppose the spherical angles describing the direction of \( \hat{p} \) relative to fixed coordinate axes are \( \theta, \phi \). Then \( \hat{g}_x \hat{p} \hat{y} = \hat{p} \hat{y} \hat{g}_x \) where

\[
\Psi = \begin{pmatrix}
\cos \phi/2 \ e^{i \theta/2} \\
\sin \phi/2 \ e^{i \theta/2}
\end{pmatrix}
\]
Then \( \psi_a = \psi \) and \( \psi = \psi_a \frac{1}{\sqrt{2E}} \left( \begin{array}{c} \sqrt{E+m} \cos \theta e^{i\phi_a} \\ \sqrt{E+m} \sin \theta e^{i\phi_a} \\ \sqrt{E-m} \sin \theta e^{-i\phi_a} \\ \sqrt{E-m} \cos \theta e^{-i\phi_a} \end{array} \right) \)

(positive helicity spinor)

and negative helicity spinor is \( \psi = \psi_a \frac{1}{\sqrt{2E}} \left( \begin{array}{c} -\sqrt{E+m} \sin \theta e^{i\phi_a} \\ +\sqrt{E+m} \cos \theta e^{i\phi_a} \\ +\sqrt{E-m} \sin \theta e^{-i\phi_a} \\ -\sqrt{E-m} \cos \theta e^{-i\phi_a} \end{array} \right) \)

let \( p \) be in \( \hat{z} \) direction. For helicity \( (h) = + \), \( \psi = \frac{1}{\sqrt{2E}} \left( \begin{array}{c} -\sqrt{E-p} \cos \theta e^{i\phi_a} \\ \sqrt{E-p} \sin \theta e^{i\phi_a} \\ \sqrt{E+p} \sin \theta e^{-i\phi_a} \\ \sqrt{E+p} \cos \theta e^{-i\phi_a} \end{array} \right) \)

(positive helicity spinor)

and negative helicity spinor is \( \psi = \psi_a \frac{1}{\sqrt{2E}} \left( \begin{array}{c} \sqrt{E-p} \sin \theta e^{i\phi_a} \\ -\sqrt{E-p} \cos \theta e^{i\phi_a} \\ -\sqrt{E+p} \sin \theta e^{-i\phi_a} \\ \sqrt{E+p} \cos \theta e^{-i\phi_a} \end{array} \right) \)

For \( E = -E_p \) in a "vacuum" Dirac proposed that the negative energy states exist, but are all filled with negative electrons. This invites the interesting possibility that a photon will excite an electron from a negative energy state to a positive energy state. The appearance of an electron with positive energy and the absence of an electron with negative energy from the negative energy sea of of electrons is physically the same as the creation of an electron and a positron, both with positive energy.

Example of the use of Dirac Equation — Scattering in a Coulomb Potential (first Born approximation)

The scattering cross section is given by

\[
6 \sigma = 2 \pi \delta(E_f - E_i) \frac{d^3 \mathbf{p}_f}{d^3 \mathbf{p}_i} |M_{lf}|^2
\]
\[ M_{fi} = \int \psi_+^* V(x) \psi_i(x) \, dx \]

\[ \psi_+ (x) = e^{i(B - A) \cdot x - E_+ t} u \]

\[ M = \int u_+^* \frac{e^{i(B - A') \cdot x - E_+ t}}{r} \frac{Z e^2}{\sqrt{r}} \, u_1 \, dx \]

\[ = (u_+^* u_1) \frac{4\pi Z e^2}{Q^2} \quad \delta = p_1 - p_2 \]

\[ \delta(E_+ - E_1) \] insures that \( |p_1| = |p_2| = p \)

then,

\[ 6 \, \nu_1 = \frac{2\pi}{(2\pi)^3} \int \frac{dp^*}{(2\pi)^3} \delta (\sqrt{p^*_+ m^2} - \sqrt{p_1^*_+ m^2}) \rho^2 \, dp^* \left( \frac{4\pi Z e^2}{Q^2} \right) \text{Re} (u_+^* u_1) \]

\[ = (2\pi)^{-2} \rho \left( \frac{16\pi^2 Z e^2}{Q^2} \right) \, \frac{E_+}{\rho} \, dS \, \frac{1}{Q^2} \left| (u_+^* u_1) \right|^2 \]

\[ 6 \nu_1 = \frac{4Z e^2 (m^2 + p^2) |u_+^* u_1|^2}{16 p^+ \sin^2 \theta/2} \]

We must still evaluate \( u_+^* u_1 \).

There are 4 cases:

- \( h = + \) \quad \rightarrow \quad \lambda_+ \text{ or } \lambda = -

- \( h = - \) \quad \rightarrow \quad \lambda = + \text{ or } \lambda = -

\[ u_1 : \quad h = + \quad u_1 = \left( \frac{\sqrt{E+m}}{\sqrt{E-m}} \right) \frac{1}{V_{2E}} \]

\[ u_2 : \quad h = + \quad u_2 = \frac{1}{V_{2E}} \left( \frac{\sqrt{E+m} \cos \theta/2}{\sqrt{E+m} \sin \theta/2} \right) \quad \lambda = - \]

\[ u_3 : \quad h = + \quad u_3 = \frac{1}{V_{2E}} \left( \frac{\sqrt{E+m} \sin \theta/2}{\sqrt{E-m} \cos \theta/2} \right) \]

\[ u_4 : \quad h = + \quad u_4 = \frac{1}{V_{2E}} \left( \frac{\sqrt{E+m} \sin \theta/2}{\sqrt{E-m} \cos \theta/2} \right) \]
\[
\begin{align*}
h=+ & \rightarrow h=+, \quad u_u^+ u_u = \cos \Theta/2 \\
\bar{h}=+ & \rightarrow \bar{h}=-, \quad u_n^+ u_n = -\frac{m}{E} \sin \Theta/2 \\
\bar{h}=- & \rightarrow \bar{h}=-, \quad u_n^+ u_n = \cos \Theta/2 \\
\bar{h}=- & \rightarrow \bar{h}=+, \quad u_n^+ u_n = \frac{M}{E} \sin \Theta/2
\end{align*}
\]

If the helicity of the incoming electron is +, then the scattering cross section is

\[
\begin{align*}
\frac{Z^2 e^4 (m^2 p^2)}{4 p^4 \sin^2 \Theta} \left( \frac{E^2 + m^2 \sin^2 \Theta}{2} + m^2 \sin^2 \Theta \right) \\
= \frac{Z^2 e^4}{4 p^4 \sin^2 \Theta} \left( E^2 \cos^2 \Theta \frac{1}{2} + m^2 \sin^2 \Theta \right) \\
= \frac{Z^2 e^4}{4 p^4 \sin^2 \Theta} \left( 1 + p^2 \cos^2 \Theta \right)
\end{align*}
\]

The cross-section is the same with incoming electron of \( \bar{h} = - \), and also the same as the cross-section for scattering unpolarized electrons.

**Problem:** Calculate the cross-section for the Compton effect using the solutions of the Dirac equation for a free particle and perturbation theory.

**Compton effect:** incoming photon \( k_1 \), \( \rightarrow \) outgoing photon \( k_2 \), incoming electron \( p_1 \), \( \rightarrow \) outgoing electron \( p_2 \)
CALCULATION OF THE COMPTON EFFECT

\[ \gamma + e^- \rightarrow \gamma + e^- \]

(notice that the process \( \gamma + e^- \rightarrow e^- \) can't be a real process, because it is impossible to conserve energy and momentum. Hence Compton scattering is the simplest process involving electrons and photons)

Choose to work this in the lab system

\[
\begin{array}{c}
\begin{aligned}
\gamma & \rightarrow \gamma \\
\gamma & \rightarrow \gamma
\end{aligned}
\end{array}
\]

(wiggly lines are photons; straight ones are electrons)

Write down all the 4 vectors \((A_0, A_x, A_y, A_z)\) associated with the problem:

\[
P_1 = (m, \omega, \theta, 0) \quad \omega = \frac{\gamma}{\gamma - \omega} \quad P_2 = (E_2 - \omega, 0, 0, \omega, \theta)
\]

\[
P_2 = (E_2 - \omega, 0, 0, \omega, \theta)
\]

Final state quantities are constrained by

\[
\frac{P_1}{P_2} + \frac{h_1}{h_2} = \frac{P_2}{P_2} + h_2
\]

energy-momentum conservation

\[
\frac{P_2}{P_2} = m^2 = (P_1 + h_1 - h_2) = m^2 + 2 \rho \cdot h - 2 \rho \cdot h_2 = 2 h, h
\]

This gives the well known formula for change of \(\gamma\) ray frequency

\[
\kappa = 1 - \omega \cdot \theta = \frac{m}{\omega - \omega'} - \frac{m}{\omega}
\]

Must also consider the possible polarizations of the \(\gamma\) ray

\[-\quad \text{plane of scattering} \quad (0, 0, 1, 0)
\]

\[\|\quad \text{plane of scattering} \quad \text{possible for both} \quad \gamma_1 \quad \text{and} \quad \gamma_2
\]

use \(e \cdot h = 0\) to get possibilities

\[
\begin{align*}
\gamma_1 &= (0, 1, 0, 0) \\
\gamma_2 &= (0, \omega, \theta, 0, - \sin \theta)
\end{align*}
\]

The cross section formula we are about to derive was first obtained by Klein and Nishina. Then they did it, it was considered a difficult and complicated problem; with the computational tricks to be taught in this course plus a little practice the average Ph 205 student will be able to complete problems of this
difficulty in less than \( \frac{1}{2} \) hour.

\[
\text{Probability of transition/sec} = 2\pi \int \left( E_{\omega} - E_{\nu} \right) \frac{d^3 k_2}{(2\pi)^3} |M|^2
\]

(Notice: this formula always holds, relativistic kinematics or no relativistic kinematics; it is the change in the way you write \( \mathbf{E} \) inside the \( \int \) function which changes the ultimate form for the phase space)

\[
\mathcal{H} = \beta m + \mathbf{\hat{P}} \cdot \left( \mathbf{\hat{P}} - \frac{e}{c} \mathbf{\hat{A}} \right) = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \quad \mathcal{H}_{\text{int}} = -\frac{e}{c} \mathbf{\hat{A}}
\]

(notice that Dirac theory contains no \( \mathbf{\hat{A}}, \mathbf{\hat{A}}^\dagger \) terms)

\[
\mathbf{\hat{A}} = \sum_k \frac{1}{\sqrt{2w}} \left[ a_k \hat{c}_k e^{i\mathbf{\hat{k}} \cdot \mathbf{\hat{r}}} + a_k^\dagger \hat{c}_k^\dagger e^{-i\mathbf{\hat{k}} \cdot \mathbf{\hat{r}}} \right]
\]

sum is over 2 different possible polarizations for each \( k \)

Since Compton scattering requires the annihilation of one photon and the creation of another, the first contribution in perturbation theory comes from second order.

\[
\text{Prob/sec} = 2 \pi \int \left( \frac{m^2 - \left( k_1^2 - k_2^2 \right)^2 + \omega_2 - m - \omega_1}{(2\pi)^3} \right) \frac{k_2^2}{(2\pi)^3} d\mathcal{L}_2 |M|^2
\]

If we cancel the \( \int \) function against \( d\mathcal{L}_2 \) we get an angular distribution for the emitted photons; if we cancel it against \( d\mathcal{L}_2 \) we get a frequency spectrum (there are of course related by *)

Generally one measures angular distributions.

\[
\delta(f(x)) = \frac{\delta(x-a)}{|f'(a)|}
\]

if \( f = 0 \) at \( x = a \)

Here

\[
f(k_2) = \frac{\omega_2 - \omega_1}{E_2 - m - \omega_1} - \omega_1 + \omega_1
\]

\[
f'(k_2) = \frac{\omega_2 - \omega_1 \omega_2 \theta}{E_2} + 1
\]

This must be evaluated at

\[
f'(a) = \frac{\omega_2 - \omega_1}{E_2}
\]

Thus

\[
\text{Prob/sec} = \sigma \mathcal{C} = \frac{\omega_2^2}{m m_1} \frac{E_2}{(2\pi)^2} d\mathcal{L}_2 |M|^2 = \text{cross section} \times \text{relative velocity}
\]

where

\[
E_2 = m + \omega_1 - \omega_2 \quad \text{and} \quad 1 - \cos \theta = \frac{m}{\omega_2} - \frac{m}{\omega_1}
\]

allow one to get the whole mess solely in terms of the angles and incident energy

Now for the matrix element: From first term we recall that there are two contributions to the second order matrix element.

At first annihilate \( \phi_1 \), then create \( \phi_2 \).

\[
\mathcal{A} = \sum_i \left< 2 \left| \phi_2 \phi_2^* e^{i\mathbf{k}_2 \cdot \mathbf{r}_2} \phi_1 \phi_1^* e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} \right| 1 \right>
\]

\[
\frac{E_1 + \omega_1 - E_2}{E_1 + \omega_1 - E_2}
\]
\[ \sqrt{\frac{2\omega_k}{\hbar^2}} = \sum_j \left( \frac{<\alpha_j | e^{-i\hbar k \cdot x} \beta_j>}{E_1 + \omega_1 - (E_j + \omega_1 + \omega_2)} \right) \]

\[ |E_i| = \sqrt{p_i^2 + m^2} \]

from the way we got the formula

All that remains is to stick in the wave functions and get the numbers. Each electron wave function has the form \( u e^{i k \cdot x} \), where u is a 4 component spinor. Everyone knows what to do with the \( e^{i k \cdot x} \) pieces (integral over x just gives momentum conservation) so we forget about this for now and concentrate on the matrix part.

Each of the sums must be split into a sum over positive energy intermediate states and over negative energy states

\[ A = \sum_{j, \text{positive energy}} \frac{<\alpha_j | d_j | u_i> <\alpha_j | d_j | u_i>}{E_1 + \omega_1 - (E_j + \omega_1 - i\varepsilon)} + \sum_{j, \text{negative energy}} \frac{<\alpha_j | d_j | u_i> <\alpha_j | d_j | u_i>}{E_1 + \omega_1 + (E_j + \omega_1 + i\varepsilon)} \]

\[ B = \sum_{j, \text{positive energy}} \frac{<\alpha_j | d_j | u_i> <\alpha_j | d_j | u_i>}{E_1 - \omega_2 - (E_j - i\varepsilon)} + \sum_{j, \text{negative energy}} \frac{<\alpha_j | d_j | u_i> <\alpha_j | d_j | u_i>}{E_1 - \omega_2 + (E_j + i\varepsilon)} \]

Keep in mind that the states \( j \) have different momentum from those labelled \( i \).

Physically, however, we do not allow the possibility of negative energy states. We do not have the new possibility of pair creation. This means we should eliminate all sums over negative energy states and add in terms for diagrams like

\[ \text{add to } A \]

The idea expressed in terms of hole theory is as follows:

You can't knock an electron into a negative energy state, because they are full.

However, a photon can excite an electron out of one of these states, leaving a hole in the sea. Then the next photon can make one of the electrons drop back into the negative energy state.

New term to be added to B has energy denominator

\[ E_1 + \omega_1 - (E_1 + |E_k| + E_2) - i\varepsilon = - (E_1 - \omega_2 + |E_k|) - i\varepsilon \]

But overall contribution must be multiplied by a - sign relative to the first term in B because of the following argument.
We should really put in our matrix element the grand wave function of all electrons in the world, even though the Hamiltonian acts only on one particular electron. The incident g.w.f. is the same for both pictures. But \( B_2 \) differs from \( B_1 \) in the final state, because in \( B_1 \) the final state is the same electron as initially whereas in \( B_2 \) the initial electron has been exchanged for one in the sea. The g.w.f. is totally antisymmetric under interchange of any two electrons.

\[ \text{Hence the only overall change is in the sign of the } i \in \text{ contribution to the denominator} \]

\[ A = \sum \frac{u^*_2 \mid d_2 \mid u_i^* \rangle \langle u_i \mid d_1 \mid u_1^* \rangle}{E_{i+} + \omega_i - 1E_i - i \epsilon} + \sum \frac{u^*_2 \mid d_2 \mid u_i^* \rangle \langle u_i \mid d_1 \mid u_1^* \rangle}{E_{i+} + \omega_i + 1E_i + i \epsilon} \]

\[ B = \sum \frac{u^*_2 \mid d_1 \mid u_1^* \rangle \langle u_i \mid d_1 \mid u_i^* \rangle}{E_{i-} - \omega_i - 1E_i - i \epsilon} + \sum \frac{u^*_2 \mid d_1 \mid u_1^* \rangle \langle u_i \mid d_1 \mid u_i^* \rangle}{E_{i-} - \omega_i + 1E_i + i \epsilon} \]

When the arrow on the solid line is running backward in time you have a positron (hole).

It is possible (although difficult!) to do an experiment in which all the spins of incident and final particles are polarized. However, one must always sum over the spins of the internal states. When one contemplates just how many matrix elements have to be summed if the external particles are unpolarized, the problem that faced Klein and Moshina becomes quite clear.

Casimir invented an improvement to do the sum over intermediate states

\[ \sum_i (u^*_2 \mid d_2 \mid u_i^* \rangle \langle u_i \mid d_1 \mid u_1^* \rangle) = \sum_i (x^*)_a (u_i^*)_a (y^*)_b \]

where

\[ \lambda^+_{a \beta} = \sum_{e i n e r g y} (u_i^*)_a (y^*)_b \]

Explicit computation with the \( u \)'s shows that

\[ \lambda^+ = \frac{1}{2E} \left[ E + m \beta - \alpha \cdot \beta \right] \]

\[ \lambda^- = \frac{1}{2E} \left[ E - m \beta - \alpha \cdot \beta \right] \]

These can be obtained more simply:

\[ \sum_{\text{all states}} u_i \mid d_1 \rangle \langle u_i \mid d_1 \rangle = \frac{1}{2} \]

\[ \sum_{+ \text{energies}} u^*_2 \mid (H + 1E_i) \mid u_i \rangle \langle u_i \mid M \mid u_1 \rangle = \sum_{\text{all energies}} u^*_2 \mid (H + 1E_i) \mid u_i \rangle \langle u_i \mid M \mid u_1 \rangle \]

\[ = u^*_2 \frac{N}{2E_i} \frac{(H + 1E_i) u_i \rangle \langle u_i \mid M \mid u_1}{2} \]
Likewise \[ \mathcal{J} = \frac{H - iE \mathbf{i}}{2E \mathbf{j}} = \frac{iE \mathbf{i} - H}{2E \mathbf{i}} \]

We thus see that any sum over positive energy intermediate states is got by inserting \( \gamma^+ \); similarly for \( \gamma^- \) and negative energy intermediate states.

Hence we have

\[
A = \frac{u^+_2 \delta_2 (1E_1 + m \beta + \vec{E} \cdot \vec{p}_i) \alpha_1 \mathbf{u}_1}{2E_1 (E_1 + \mathbf{u}_1 - iE \mathbf{i})} + \frac{u^*_2 \delta_2 (1E_1 - m \beta - \vec{E} \cdot \vec{p}_i) \alpha_1 \mathbf{u}_1}{2E_1 (E_1 + \mathbf{u}_1 - iE \mathbf{i})}
\]

\[
= \frac{u^+_2 \delta_2 \left[(E_1 + \mathbf{u}_1 + m \beta + \vec{E} \cdot \vec{p}_i)^2 - E_1^2 \right]}{(E_1 + \mathbf{u}_1 - iE \mathbf{i})^2 - (\vec{E} \cdot \vec{p}_i)^2 - m^2} \beta \mathbf{u}_1
\]

Define \( u^+ \beta = \overline{\mathbf{u}} \)

Thus if \( u^+ = (u_1^+, u_2^+, u_3^+, u_4^+) \) then \( \overline{\mathbf{u}} = (u_1^*, u_2^*, -u_3^*, -u_4^*) \)

The final form for our \( A \) matrix element is then

\[
\frac{u^+_2 \delta_2 \left[(E_1 + \mathbf{u}_1 + m \beta + \vec{E} \cdot \vec{p}_i)^2 - m^2 \right]}{(E_1 + \mathbf{u}_1 - iE \mathbf{i})^2 - E_1^2} \beta \mathbf{u}_1
\]

Likewise \( B \) is

\[
\frac{u^+_2 \delta_1 \left[(E_1 - \mathbf{u}_1 - m \beta - \vec{E} \cdot \vec{p}_i)^2 - m^2 \right]}{(E_1 - \mathbf{u}_1 + iE \mathbf{i})^2 - E_1^2} \beta \mathbf{u}_1
\]

Feynman's contribution was to arrive at this stage and then figure out rules by which you could write these answers down without going through all the intermediate steps.

Aside \( \frac{\bar{1}}{2} \): Effect of two time inversions on a spin \( J \) system

\[
T^2 \left| \mathbf{S} = m \right\rangle \right. = \begin{cases} -1 & m \neq +1 \text{ in Feynman} \\ +1 & m = +1 \text{ in Feynman} \end{cases}
\]

How to find answer:

from elemen- \( \text{a) } T \left| J, m \right\rangle = \text{phase } \left| J, -m \right\rangle \)

\( \text{b) } T \left( \alpha \left| 1 \right\rangle + \beta \left| 2 \right\rangle \right) = \alpha^* T \left| 1 \right\rangle + \beta^* T \left| 2 \right\rangle \)

c) Integral spin, mass \( \neq 0 \Rightarrow \exists \text{ state } \mathcal{S} \neq 0 \)

\[
T \left| \psi \right\rangle = e^{i\delta / \hbar} \left| \psi \right\rangle
\]

\[
T^2 \left| \psi \right\rangle = e^{-i\delta / \hbar} T \left| \psi \right\rangle = \left| \psi \right\rangle
\]

d) Half integral spin

\[
T \left| + \right\rangle = e^{i\delta / \hbar} \left| + \right\rangle \quad T \left| - \right\rangle = e^{i\delta / \hbar} \left| - \right\rangle
\]

\[
\frac{1}{\sqrt{2}} T \left[ \left| + \right\rangle + \left| - \right\rangle \right] \text{ must be spin down in } x \text{ direction}
\]

\[
= \frac{e^{i\delta / \hbar}}{\sqrt{2}} \left[ \left| + \right\rangle + e^{i\delta / \hbar} \left| - \right\rangle \right] = \frac{e^{i\delta / \hbar}}{\sqrt{2}} \left[ e^{i\delta / \hbar} \left| - \right\rangle + e^{i\delta / \hbar} \left| + \right\rangle \right]
\]

\[
= \frac{\sqrt{2}}{\sqrt{2}} e^{i\delta / \hbar} \left[ \left| + \right\rangle + e^{i\delta / \hbar} \left| - \right\rangle \right] \Rightarrow e^{i(\delta - \delta')} = -1
\]
Aside §2: Because the electron has only two possible helicities, it should be possible to describe it by just a 2 component wave function, rather than a 4 component one.

One way to do this is to start with the two component wave function for an electron at rest and then transform it to the desired momentum.

For a state at rest \( |0, i\rangle \) where \( i \) describes the spin state
\[ e^{-i \frac{P \cdot \vec{v}}{m}} |0, i\rangle = |m \sinh \nu, \nu, 0, i\rangle = |3 \text{ momentum, } i\rangle \]
The general "boost" operator has the form \( e^{-i \frac{P \cdot \vec{v}}{m}} \)

Other operators work as expected
\[ \hat{P}_z |P_\rho, \rho, \rho_\rho^2, i\rangle = P_z |P_\rho, \rho, \rho_\rho^2, i\rangle \]
\[ \tanh \theta = \frac{P_t}{P_\rho} \quad \sinh \theta = \frac{P_t}{P_\rho} \]
defines the parameter \( \nu \) for the transformation
\[ P_\rho \frac{|0, i\rangle}{m/|0, i\rangle} \]
The operator \( P_t^2 - P_\rho^2 - \rho_\rho^2 - P_\rho^2 \) is an invariant for a given representation

Hence every state can be described in terms of a momentum and a spin state at rest.

The generators of the transformation can be written
\[ J_\rho = i \left[ \frac{\partial}{\partial \rho_\rho^2} - \rho_\rho^2 \frac{\partial}{\partial \rho} \right] + J_\rho \rightarrow (\text{intrinsic spin}) \]
\[ N_\rho = i \left[ \frac{\partial}{\partial P_\rho} - P_\rho \frac{\partial}{\partial \rho_\rho^2} \right] + \left[ \frac{\nu \times \rho_\rho^2}{E + m} \right] \]
The most general wave function can then be written in the form
\[ |\psi\rangle = \sum \left( a_+ |P^+\rangle + a_- |P^-\rangle \right) \]

and the most general scattering operator can be put as
\[ f(\rho_\rho^2, \rho_\rho^2) + \frac{\bar{N}_\rho}{\bar{J}_\rho} \left( \bar{P}_\rho, \bar{P}_\rho^2 \right) \]

For fun, figure out from conventional theory and/or this theory why the two component scattering amplitude operator in a scalar potential would take the form
\[ \frac{\sqrt{2M(E_\rho + m)}}{2M(E_\rho + m)} \left( \frac{\bar{P}_\rho \bar{P}_\rho^2 + i \bar{N}_\rho \bar{P}_\rho^2}{(\vec{P} \times \vec{P})} \right) \]
and in a pseudoscalar potential would look like
\[ \frac{\sqrt{2M(E_\rho + m)}}{2M(E_\rho + m)} \left( \frac{\bar{P}_\rho \bar{P}_\rho^2}{(E_\rho + m)} \right) \]

Question: If you scatter twice in a scalar potential according to the graph
\[ \text{can you prove from the amplitude resulting that electron and positron must have}\]
\[ \text{opposite intrinsic spin?} \]
Problem: Find the rate for annihilation in flight of a positron by an electron nearly at rest. Then calculate the cross-section for 
\[ e^+ + e^- \rightarrow 2\gamma \] for a position in flight.

Give the formula for the special case where the velocity of the positron is zero. Estimate the lifetime of positronium. Discuss the relative polarization of the 2 photons.

Compton Effect (continued)

We saw that in lowest order perturbation theory, the transition probability/sec for the process \( \gamma + e^- \rightarrow \gamma + e^- \) is given by

\[ \text{Rate} = 2\pi \delta (E_2 + w_2 - E_1 - w_1) \frac{d^3 p_2}{(2\pi)^3} |\mathcal{M}|^2 \]

where \( \mathcal{M} = \frac{4\pi e^2}{\sqrt{2\mu_12\mu_2}} \left( \frac{u_2 \eta_2 (p_1 + k_1 + m) \phi_1 U_1}{(p_1 + k_1)^2 - m^2} + \frac{u_2 \eta_2 (p_1 - k_2 + m) \phi_2 U_1}{(p_1 - k_2)^2 - m^2} \right) \) (A)

The term marked (A) is obtained from the sum of two terms which may be represented diagrammatically as

![Diagram A](image)

and similarly for (B):

![Diagram B](image)
Notation

\[ a_\mu b_\mu = a_x b_x - a_y b_y - a_z b_z \]

\[ \nabla_\mu = \left( \frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = (\partial_t, \partial_x, \partial_y, \partial_z) \]

\[ \gamma_\mu = (\beta, \beta \partial_x, \beta \partial_y, \beta \partial_z) \]

If \( a_\mu \) is any 4-vector, \( \mathbf{A} = a_\mu \gamma_\mu = \gamma_t a_t - \gamma_x a_x - \gamma_y a_y - \gamma_z a_z \)

\[ = \beta (a_t - \mathbf{a} \cdot \mathbf{\gamma}) \]

and notice that \( \nabla = \gamma_\mu \partial_\mu = \gamma_t \partial_t + \gamma_x \partial_x + \gamma_y \partial_y + \gamma_z \partial_z \)

\[ = \beta (\partial_t + \mathbf{a} \cdot \mathbf{\nabla}) \]

Rules:
1) \( A^2 = a_\mu a_\mu = \mathbf{a}^2 \)
2) \( a \beta \Rightarrow a_\mu = b_\mu \)
3) \( a \beta + b \partial = 2 \mathbf{a} \cdot \mathbf{b} \)
4) \( a \beta + \partial \mathbf{a} = 2a_\mu \)

Using this notation, the Dirac Equation is written

\[ (i \gamma_\mu \partial_\mu - \gamma_0 m) \psi = 0 \]

Ex. If \( e_1 \) represents pure x-polarization, \( e_1 = (0; 1 \ 0 \ 0) \)

and \( \gamma_1 = -\gamma_0 \gamma_x \)

for an electron at rest, \( \beta = m \gamma_t \).

Relativistic Invariance:

We deal with this question in two steps: First

1) Normalization of the \( \mathbf{u} \)'s.
2) Formula for rate — make relativistic invariance obvious.
Normalization:

Until now, our normalization has been $u^* u = 1$. Now $\mathcal{U}$ is the time component of a 4-vector and so the normalization is not relativistically invariant. We can correct this by noticing that $u^* u$ is an invariant, so a relativistically invariant normalization can be obtained from $u^* u = \text{cany}$. For reasons of later convenience, we will choose $u^* u = 2m$.

Then for an electron with spin and momentum in the $+z$ direction, the Dirac spinor is

$$u \left( \mathbf{p}, \mathbf{\gamma}_5 \right) = \begin{pmatrix} \sqrt{E + m} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{E - m} \end{pmatrix}$$

Then $\mathcal{U} \mathbf{u} = u^* u = 2E$, or the normalization is $2E$ particles per unit volume.

Let $M = M \sqrt{2E_1E_2} \sqrt{2w_12w_2}$

Then for Compton effect,

$$\text{Rate} = 2\pi \delta(E_1 - E_2) d^3k_2 \frac{1}{(2\pi)^3} \frac{1}{2E_12E_22w_12w_2}$$

$d^3k_2$ and $\delta(E_1 - E_2)$ appear in non-relativistic form.

We can change this into

$$\frac{\delta(E_1 - E_2) \delta^3(p_1 - p_0)}{(2\pi)^3(2\pi)^3} d^3k_2 \frac{1}{2E_12E_22w_12w_2} \frac{(2\pi)^4}{(2\pi)^3} \frac{1}{M^2} \delta^3(p_1 - p_0)$$

Now $d^3k_2$ is not a relativistic invariant, but $d^3p_2$ is, as is also $\frac{d^3p_2}{2E_2}$

To see this, use $E = \sqrt{p^2 + m^2}$ and in another coordinate system, the energy and momentum is $E'$ and $p'$, where $E'^2 = p'^2 + m^2$.
The relativistic law of transformation is (for relative motion in the direction)

\[ I^\prime = \frac{I - vE}{\sqrt{1 - v^2}} \]

\[ J^\prime = J \]

\[ K^\prime = K \]

\[ E^\prime = \frac{E - vP_y}{\sqrt{1 - v^2}} \]

\[ \frac{dP_y'}{dP_y} = \frac{1}{\sqrt{1 - v^2}} \left( 1 - \frac{vP_y}{E} \right) = \frac{E'}{E} \]

So that \( \frac{d^3I'}{E'} = \frac{d^3I}{E} = \) relativistic invariant.

\[ \text{Rate} = \frac{(2\pi)^4 \delta^4(k f + P_i)}{(2\pi)^3 2E_1} \frac{d^3P_k}{d^3k_z} \left| m \right|^2 \]

\[ \frac{k_z}{2E_1 2W_i} \]

Where \( M = (4\pi^2) \left[ \left( \frac{k_z}{E_1} \right)^2 \left( \frac{P_i + m + k_i}{P_i + k_i} \right) \right] \]

\[ \left( \frac{(P_i + k_i)^2}{P_i + k_i} \right) \frac{(P_i + k_i)^2}{P_i + k_i} \]

\[ \left( \frac{(P_i + k_i)^2}{P_i + k_i} \right) \frac{(P_i + k_i)^2}{P_i + k_i} \]

\[ \text{relativistic invariant} \]

In general, \( M = \) relativistically invariant amplitude. To see that this is possible, we must exhibit the formula for the rate in a relativistically covariant manner. To do this we define the cross-section for a process with 2 incoming particles by with parallel (or antiparallel) velocities by

\[ \sigma \text{ rel.} = \text{Rate} \]

\[ \text{where } \sigma \text{ rel.} = \text{relative velocity of the two particles} \]

and specify that the cross-section \( \sigma \) is a relativistic invariant.

Then \( \sigma = \frac{1}{\sigma \text{ rel.} E_i W_i} \left| M \right|^2 \)

\[ \text{relativistically invariant} \]

In order to ensure that \( \sigma \) is a relativistic invariant, we must find a Lorentz scalar that reduces to \( \sigma \text{ rel.} E_i W_i \) for the case of Compton scattering with parallel velocities. We will then generalize the result to any scattering process in which there are two particles in the initial state.
Consider the scattering $1+2 \rightarrow \text{anything}$, and let
\[ B = \left[ -\frac{1}{2} \left( k_1 \cdot p_2 - k_2 \cdot p_1 \right) \left( k_1 \cdot p_2 - k_2 \cdot p_1 \right) \right]^{\frac{1}{2}} \]
which is clearly a Lorentz scalar.

Particle 1 has four momentum $k_1$, $k_1^2 = m_1^2$
Particle 2 has four-momentum $p_1$, $p_1^2 = m_1^2$

Suppose we evaluate $B$ in a coordinate system in which particle 1 and particle 2 have parallel velocities.

Then
\[ B = \left[ -m_1^2 m_2^2 + (E_1 E_2 - p_1 p_2)^2 \right]^{\frac{1}{2}} \]
\[ = \left[ 2p_1^2 p_2^2 - p_1^2 m_2^2 - p_2^2 m_1^2 + 2E_1 p_1 p_2 \right]^{\frac{1}{2}} \]
\[ = E_1 E_2 \left[ 2v_1 v_2 - v_1^2 v_2^2 - v_1^2 - v_2^2 - v_1 v_2 - 2v_1 v_2 \right]^{\frac{1}{2}} \]
\[ = E_1 E_2 |v_1 - v_2| \]

Similarly, it is easy to check, for antiparallel velocities, $B = E_1 E_2 (v_1 + v_2)$

\[ 6 = \frac{\frac{d^3 \nu}{4B} \frac{d^3 \nu}{(2\pi)^2 E_1^2 (2\pi)^2 E_2^2} \frac{1}{M^2}}{4} \]
and the expression for the cross-section is manifestly covariant.

We showed that $B = \nu \text{rel} E_1 E_2$ for 2 incoming particles in a coordinate system in which the particles have parallel velocities. Since $B$ is a Lorentz scalar, we can extend the definition of $6$ in a relativistically invariant manner to cases in which the 2 incoming particles do not have velocities in the same direction by use of the formula above.
Another way of writing $\frac{d^3p}{2E}$ is $d^3p \delta(p^2-m^2) \Theta(p)$

where $\Theta(p) = \begin{cases} 1 & p > 0 \\ 0 & p < 0 \end{cases}$ is the step function.

To save writing, the $\Theta(p)$ is usually omitted (but it is understood to be there).

For an arbitrary number of incoming particles, we then write the rate as

$$\text{Rate} = (2\pi)^4 \delta^4(p_1 - p_i) \prod_{i=1}^{n_{\text{in}}} \frac{1}{2E_{\text{in}}^i} \prod_{i=1}^{n_{\text{out}}} d^3p_i |M_{fi}|^2$$

This may also then be written,

$$\text{Rate} = \prod_{i=1}^{n_{\text{in}}} \frac{1}{2E_{\text{in}}^i} (2\pi)^4 \delta^4(p_1 - p_i) \prod_{i=1}^{n_{\text{out}}} \frac{1}{2E_{\text{out}}^i} \delta(f_j - m_j^2) d^4p_j |M_{fi}|^2$$

Shorthand may also be used in calculating perturbation theory diagrams in electrodynamics.

1) **Electron propagator**

$$\begin{array}{c}
\frac{1}{p^2 - m^2} \\
\end{array}
\text{propagator} = \frac{p + m}{p^2 - m^2} \quad (\frac{1}{p - m})$$

2) **Amplitude to interact with a photon of polarization** $\epsilon_{\gamma} \to \gamma \epsilon_{\gamma} \phi$

3) **Amplitude for**

3) **Each free electron that enters introduces a factor** $u_p$

$$\text{leaves with momentum } p, \text{ introduces a factor } u_p$$

Recall $p\cdot u = m u$. 

There is a way to remember the propagator.

Consider \((i\nabla - m) \psi = \delta = \text{source}\)

Take Fourier transform of the equation to momentum space

\[ (q^2 - m) \psi_q = \delta \]

or, \[ \psi_q = \frac{1}{q^2 - m + i\varepsilon} \delta \]

\(5\) familiar boundary condition

Now for Dirac Eq. for a particle in an electromagnetic field,

\[ S' = e A \psi \]

\[ (i\nabla - e A - m) \psi = 0 \]

in zeroth order \( \psi = \phi \)

in first order, \[ \psi = \phi + \frac{1}{q^2 - m + i\varepsilon} e A \phi \]

in higher orders,

\[ \psi = \phi + \frac{e}{q^2 - m + i\varepsilon} A \phi + \frac{e^2}{q^2 - m + i\varepsilon} \frac{1}{q^2 - m + i\varepsilon} A A \phi + \cdots \]

in coordinate space, this is written

\[ \psi = \phi + \frac{e}{i\nabla^2 - m + i\varepsilon} A \phi + \frac{e^2}{i\nabla^2 - m + i\varepsilon} \frac{1}{i\nabla^2 - m + i\varepsilon} A A \phi + \cdots \]
Gauge Invariance and Relativity

We've been working in the gauge $\phi = 0, \nabla A = 0$

with $A_\mu = e_\mu e^{-ikx}$, $e.e = -1$ (e is the polarization vector)

these conditions read $E_t = 0, k \cdot e = 0$

These conditions are not manifestly covariant, so we go to their relativistic generalization — viz., $e.e = -1, e \cdot k = 0$

The reason we have to worry about this is that if one observer uses $E_t = 0$, then the vector-potential polarization vector obtained from this one by going to another coordinate system will have $E'_t \neq 0$.

If, however, the scheme we've proposed is relativistically invariant, an observer in the second coordinate system must be able to use $E'_t = 0$, that is, in fact, do so is a consequence of gauge invariance.

Let $A'_\mu = A_\mu + \nabla_\mu \xi$.

Then gauge invariance (GI) states that the same physical results obtained with $A'_\mu$ as with $A_\mu$.

Consider

\[(i\partial - e \vec{A} - m) \psi = 0 \quad (1)\]

and \[(i\partial - e \vec{A}' - m) \psi'' = 0. \quad (2)\]

We assert that $\psi''$ differs from $\psi$ only by a phase factor, and therefore represents the same physical solution.

Let $\psi'' = e^{-i2\phi} \psi$

then \[(i\partial - e \vec{A}'' + e\nabla \xi - m) \psi = 0 \quad \text{from (2)}\]

or, \[(i\partial - e \vec{A} - m) \psi = 0 \quad \text{which is (1), which proves the result.}\]

Now let $A_\mu = e_\mu e^{-ikx}$ and $\xi = -e e^{-ikx}$
Then $A'' = (e\mu + ik_p) e^{-ik\cdot x}$  
$\quad e'' = e\mu + ik_p$

To obtain $e'' = 0$, simply choose $\lambda = \frac{e\cdot x}{e\cdot x} = \frac{e\cdot x}{e\cdot x}$.

Our conditions are  
$\quad e\cdot e = -1$
$\quad e\cdot k = 0$

and these conditions hold for $e''$ also  
$\quad e''\cdot k = e\cdot k = 0$  since $k\cdot k = 0$
$\quad e''\cdot e'' = e\cdot e = -1$  since $e\cdot k = 0$

If we tried to put $x = \beta e^{ik\cdot x}$ the conditions would not be satisfied unless $k' = \pm k$.

All results must therefore be invariant under $e\mu = e\mu + ik_p$.

Let us see how this works in Compton scattering in lowest order:

$$\frac{d\sigma}{d\Omega} = \frac{M}{4\pi e^2} = \frac{\bar{u}_2 g_2 (p_1 + k_1 + m) u_1 + \bar{u}_2 g_2 (p_1 - k_1 - m) u_1}{(p_1 + k_1)^2 - m^2}$$

Let $e_1 \rightarrow e_1' = e_1 + \beta k_1$

Then $M \rightarrow M + \beta \bar{u}_2 g_2 (p_1 + k_1 + m) u_1 + \beta \bar{u}_2 g_2 (p_1 - k_1 - m) u_1$  \hspace{1cm} $\text{A}$

\hspace{1cm} $\frac{(p_1 + k_1)^2 - m^2}{(p_1 - k_1)^2 - m^2}$

\hspace{1cm} $\text{B}$

$\bar{u}_2 g_2 (p_1 + k_1 + m) u_1 = \beta \bar{u}_2 g_2 u_1$  \hspace{1cm} $\text{A}$

$\bar{u}_2 g_2 (p_1 - k_1 - m) u_1 = \beta \bar{u}_2 g_2 u_1$  \hspace{1cm} $\text{B}$

Since $p_1 + m = m$

$\bar{u}_2 g_2 (p_1 + k_1 + m) u_1 = \beta \bar{u}_2 g_2 (p_1 - k_1 - m) u_1$

$\bar{u}_2 g_2 (p_1 + k_1 + m) u_1 = -\bar{u}_2 g_2 (p_1 - k_1 - m) u_1$

$\text{A} + \text{B} = 0$

$\quad$ and GI holds.

Aside:

Arguments based on one diagram being bigger than another in some process require a specification of the gauge being used.
In this section we will work out the matrix elements from standard nonrelativistic perturbation theory, using all the field theory learned to date. Despite the fact that none of the intermediate steps look relativistically invariant, the answer will appear in a relativistically invariant form. We will then formulate rules for writing down the relativistically invariant form at once from the diagrams, and will never again work through the mess that is displayed below.

\[ 
\mathcal{H} = \frac{1}{8\pi} \int \left( E^2 + B^2 \right) d^4v + \int \psi^\dagger(x) \left[ \beta \gamma^\mu \gamma^\nu \sigma^{\nu\mu} \gamma^\nu + \frac{e}{c} \left( \frac{\vec{A}}{c} \cdot \nabla \right) \right] \psi(x) d^3x 
\]

Energy of Free Electromagnetic Field

where the \( \Psi \)'s are operators which create and annihilate electrons

This may be broken down further into the energy of the free electron-positron field

\[ \int \psi^\dagger(x) \left[ \beta \gamma^\mu \right] \psi(x) d^3x \]

and a term which represents the interaction between the electric field and the matter fields

\[ e \int \psi^\dagger(x) \left[ \sigma^{\nu\mu} \gamma^\nu \right] \psi(x) d^3x = \int \frac{e}{c} \mathcal{J}_\mu(x) A_\mu(x) d^3x \]

where

\[ \mathcal{J}_\mu = e \psi^\dagger \gamma_\mu \psi = e \overline{\psi}(x) \gamma_\mu \psi(x) \]

and it is important to keep in mind that each of \( \Psi, \Psi^\dagger, A_\mu \) is linear in the appropriate creation operators.

Now we ruin the manifest covariance in order to relate pieces to previously learned physics.

Choose a gauge such that

\[ \nabla \cdot \vec{A} = 0 \quad \nabla^2 \phi = \rho \]

(a point charge has \( \phi(r') = \frac{e}{\epsilon_0} \int \frac{\rho(r)}{|r' - r|} d^3r \)

\[ \mathcal{J}_0(x) = \rho = e \overline{\psi}(x) \gamma_0 \psi(x) \]

Thus the \( e \int \psi^\dagger \phi(x) \psi(x) d^3x \) term can be written

\[ \frac{e^2}{2} \int \psi^\dagger(x) \psi(x) \psi^\dagger(y) \psi(y) \frac{d^3x d^3y}{r_{xy}^2} \]

and we have managed to express this piece of the interaction without use of photon creation and annihilation operators. The \( \infty \) mentioned above comes from the \( \mathcal{J} \) function in the anti-commutation relations for \( \psi^\dagger(x) \) and \( \psi(y) \),

It is a self energy term, of the type discussed before in this course. The thing to notice here is that this infinity seems to involve only spatial coordinates and
thus is not obviously relativistically covariant (in another system the self-energy subtracted might be different). This was one of the difficulties of formulating the theory this way. Later on in this course the removal of certain relativistically covariant infinities will be discussed.

Then the interaction of the electron-positron field with photons is given by

\[ e \int \mathcal{L}(x) \bar{A}(x) \cdot \mathcal{A}(x) \, d^3x \]

where \( \mathcal{A}(x) = \sum \frac{\epsilon_{\nu} \cdot \mathbf{r}}{2\omega} \left( e^{i\mathbf{k} \cdot \mathbf{x}} + e^{i\mathbf{q} \cdot \mathbf{x}} \right) \epsilon_{\alpha \beta} \epsilon_{\gamma} \)

where \( \epsilon_{\alpha \beta} \) is the polarization vector for a photon of polarization type \( i \) (helicity \( \pm 1 \)) and \( \mathcal{L}(x) = \sum_{\rho, \lambda} \mathcal{L}_{\rho, \lambda} \cdot \mathbf{c}_{\rho, \lambda} \cdot e^{i\mathbf{r} \cdot \mathbf{x}} \)

\( \mathcal{L}_{\rho, \lambda} \) is a 4-component spinor for a solution of the Dirac equation of momentum \( p \), type \( j \) (\( \mathcal{L} \) runs over 4 possibilities - positive energy spin up and spin down, and negative energy spin up and spin down)

\[ (\beta \mathbf{p} + \mathbf{A} \cdot \mathbf{p}) \cdot \mathcal{L}_{\rho, \lambda} = \mathcal{E} \mathcal{L}_{\rho, \lambda} \]

\( \mathcal{E} = \pm \mathcal{E}_+ = \pm \sqrt{m^2 + \mathbf{p}^2} \)

Likewise \( \mathcal{L}^*_{\rho, \lambda} = \sum \mathcal{L}_{\rho, \lambda} \cdot \mathbf{c}_{\rho, \lambda} \cdot e^{-i\mathbf{r} \cdot \mathbf{x}} \)

Change notation:

let \( \mathcal{L}_{\rho, \lambda} \) stand for those solutions such that \( \mathcal{E} = + \mathcal{E}_{\rho, \lambda} \)

where \( \lambda \) is a spin index with two possible values

\( \mathcal{L}_{\rho, \lambda} = \mathcal{L}_{\rho, \lambda} \) for those solutions with \( \mathcal{E} = + \mathcal{E}_{\rho} \)

Then we have

\[ (\beta \mathbf{p} + \mathbf{A} \cdot \mathbf{p}) \cdot \mathcal{L}_{\rho, \lambda} = \mathcal{E} \mathcal{L}_{\rho, \lambda} \]

And the free Hamiltonian takes the form

\[ H_{el, \rho, \lambda} = \sum_{\rho, \lambda} \mathcal{E}_{\rho} \mathbf{c}_{\rho, \lambda} \mathbf{c}_{\rho, \lambda}^* = \sum_{\rho, \lambda} \mathcal{E}_{\rho} \left[ \mathbf{c}_{\rho, \lambda} \mathbf{c}_{\rho, \lambda}^* - \mathbf{c}_{\rho, \lambda}^* \mathbf{c}_{\rho, \lambda} \right] \]

where \( \mathbf{c}_{\rho, \lambda} \), \( \mathbf{c}_{\rho, \lambda}^* + \mathbf{c}_{\rho, \lambda}^* \mathbf{c}_{\rho, \lambda} = \delta_{\rho, \rho'} \delta_{\lambda, \lambda'} \)

In accordance with our new terminology for the spinors, define a new terminology for the annihilation operators of negative energy states

If \( \mathbf{c}_{\rho, \lambda} \) annihilates an electron of momentum \( p \), spin type \( \lambda \), then \( \mathbf{d}_{\rho, \lambda} \) creates a positron with momentum \( -p \).

Then

\[ H_{free} = \sum_{\rho} \mathcal{E}_{\rho} \left[ \mathbf{c}_{\rho} \mathbf{c}_{\rho}^* + \mathbf{d}_{\rho} \mathbf{d}_{\rho}^* \right] - \sum_{\rho, \lambda} \mathcal{E}_{\rho} \]
This is the energy of the full sea of negative energy states. We measure relative energies away from it.

Now \( \Psi = \sum_{\rho, \mu} \left[ c_{\rho, \mu} u_{\rho, \mu} e^{i \rho \cdot x} + d_{\rho, \mu}^* v_{\rho, \mu} e^{-i \rho \cdot x} \right] \)

notice that creation of positron in initial state (i.e. annihilation of positron in final state) gives a column vector \( v_{\rho, \mu} \) whereas annihilation of electron in final state (by \( \Psi^+ \)) gives a row vector \( u_{\rho, \mu} \).

Hence a positron in the initial state leads to \( \hat{\mathcal{U}} \) on the left side of the matrix element, whereas an electron in the initial state leads to \( \hat{\mathcal{U}} \) on the right side of the matrix element.

Use this to expand out the interaction Hamiltonian

\[
\hat{A} = \sum_{\mu} \int \frac{d^3 \chi}{2 w_{\mu}} \left[ \vec{e}_{\mu}^+ \cdot \vec{a}_{\mu} e^{i \hbar \cdot x} + \vec{e}_{\mu}^+ \cdot \vec{a}_{\mu} e^{-i \hbar \cdot x} \right]
\]

\[
\Psi = \sum_{\rho, \mu} \left[ c_{\rho, \mu} u_{\rho, \mu} e^{i \rho \cdot x} + d_{\rho, \mu}^* v_{\rho, \mu} e^{-i \rho \cdot x} \right]
\]

to obtain

\[
\sum_{\lambda} \frac{1}{\sqrt{2} w_{\lambda}} \int \left[ \int \left[ c_{\rho, \mu}^* u_{\rho, \mu} e^{-i \rho \cdot x} + d_{\rho, \mu}^* v_{\rho, \mu} e^{i \rho \cdot x} \right] \left[ \vec{e}_{\mu}^+ \cdot \vec{a}_{\mu} e^{i \hbar \cdot x} + \vec{e}_{\mu}^+ \cdot \vec{a}_{\mu} e^{-i \hbar \cdot x} \right] \right]
\]

In every case the spatial integration will give only a \( \int \) function of the momenta (because the exponentials are the only functions of \( x \) present).

We then obtain a sum of the following pieces:

\[
\sqrt{\frac{4 \pi e^2}{2 w_{\lambda}}} \cdot c_{\rho, \mu}^* a_{\lambda} \cdot c_{\rho', \mu'} \cdot u_{\rho, \mu}^* \vec{e}_{\mu} \cdot u_{\rho', \mu'} \cdot \int (\rho + \lambda - \rho')
\]

\[
\sqrt{\frac{4 \pi e^2}{2 w_{\lambda}}} \cdot c_{\rho, \mu}^* d_{\rho', \mu'} \cdot u_{\rho, \mu}^* \vec{e}_{\mu} \cdot v_{\rho', \mu'} \cdot \int (\rho - \lambda - \rho')
\]

\[
\sqrt{\frac{4 \pi e^2}{2 w_{\lambda}}} \cdot c_{\rho, \mu}^* a_{\lambda} \cdot c_{\rho', \mu'} \cdot u_{\rho, \mu}^* \vec{e}_{\mu} \cdot u_{\rho', \mu'} \cdot \int (\rho + \lambda + \rho')
\]

\[
\sqrt{\frac{4 \pi e^2}{2 w_{\lambda}}} \cdot c_{\rho, \mu}^* d_{\rho', \mu'} \cdot u_{\rho, \mu}^* \vec{e}_{\mu} \cdot v_{\rho', \mu'} \cdot \int (\rho + \lambda - \rho')
\]
The arrows show which is the entrance state and which the exit state in the matrix element. In writing a matrix element down from a picture start at the beginning of the arrow and the right of the matrix element; then follow the arrows and move to the left in the matrix element, putting in interaction matrices and spinors as they arise. Any line which points backwards in time belongs to a positron; those which point forward in time belong to electrons (It is best not to put arrows on boson lines except possibly to remind yourself of the sign of the momentum).

The 16 terms which arise from the Coulomb interaction can also be expressed in terms of pictures, even though the intermediate lines weren't obtained by explicit creation and annihilation operators.

For fun and practice with the operator anti-commutation relations, see what the equation of motion \( i \frac{\partial \Psi}{\partial t} = \mathcal{H} \Psi - \Psi \mathcal{H} \) tells you about the operator \( \gamma \).
Previously we obtained the electron propagator to be \( \frac{1}{p - m} \).

It should be possible to obtain the photon propagator from the pieces of matrix elements above.

Consider scattering to order \( \epsilon^2 \)

\[
\left< \frac{A_3}{\epsilon_3} \right| \left< \frac{A_4}{\epsilon_4} \right>
\]

for \( A = \gamma_3 - \gamma_1 \)

\[ q_{\mu} = (\gamma_{\epsilon_3}, Q_3, 0, 0) \) the contributions are

1) Coulomb interaction

\[
\frac{\alpha e^2}{2 \omega_4} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

2) Virtual transverse photons

\[
\frac{\alpha e^2}{2 \omega_4} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

\[
\sum_{\text{internal polarizations}} \frac{\alpha e^2}{2 \omega_4} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

\[
\frac{2}{2 \omega_4} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

Notice that if there were 3 directions of polarization (two \( \perp \vec{q} \) and one \( \parallel \vec{q} \)),

then \( \sum_{\text{all 3} i} \left( \hat{A} \cdot \hat{e}_i \right) \left( \hat{B} \cdot \hat{e}_i \right) = \vec{A} \cdot \vec{B} \)

Hence \( \sum_{\text{transverse}} \left( \hat{A} \cdot \hat{e}_i \right) \left( \hat{B} \cdot \hat{e}_i \right) = \vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{q} \vec{B} \cdot \vec{q} \)

Thus combination of the two transversely polarized pieces gives

\[
\left( \frac{\alpha e^2}{2 \omega_4} \right) \left[ \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right) - \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right) \right]
\]

\[
\left[ \frac{1}{E_1 - E_3 - \omega_4 + i \epsilon} + \frac{1}{E_2 - E_4 - \omega_4 + i \epsilon} \right]
\]

But \( \frac{1}{E_1 - E_3 - \omega_4} + \frac{1}{E_2 - E_4 - \omega_4} = \frac{2 \omega_4}{Q^2} \)

Hence the sum of all these terms gives

\[
\frac{\alpha e^2}{2} \left( \frac{1}{Q^2} \right) \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right) + \frac{1}{Q^2} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

\[
- \frac{1}{Q^2} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

But \( q_{\mu} u_{\gamma} \gamma_{\mu} u_{\gamma} = \gamma_{\epsilon_4} \left( \gamma_{\epsilon_4} - \gamma_{\epsilon_3} \right) u_{\gamma} = \gamma_{\epsilon_4} \left( m - m \right) u_{\gamma} = 0 \)

(current conservation)

Hence \( \gamma_{\epsilon_4} \left( \gamma_{\epsilon_4} - \gamma_{\epsilon_3} \right) u_{\gamma} = q_{\epsilon_4} \gamma_{\epsilon_4} \gamma_{\epsilon_4} u_{\gamma} \)

and we arrive at

\[
\frac{-1}{Q^2} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right) + \frac{1}{Q^2} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right) + \frac{1}{Q^2} \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)
\]

\[
- \frac{q_{\epsilon_4}^2 \left( u_{\gamma} G_{\gamma}^+ u_{\gamma} \right) \left( u_{\nu} \gamma^\mu u_{\nu} \right)}{Q^2}
\]

But \( \frac{1}{Q^2} + \frac{1}{Q^2} - \frac{q_{\epsilon_4}^2}{Q^2} = 0 \)
Hence we have shown that all the lowest order interactions due to charge coalesce into one manifestly covariant matrix element

\[ \frac{-4\pi e^2}{q^2} \left( \bar{U}_{1\mu} \gamma_\mu U_{2\mu} \right) \left( \bar{U}_{3\mu} \gamma_\mu U_{1\mu} \right) \]

which can be symbolized by the picture

\[ \begin{array}{c}
\text{\mu} \\
\downarrow \\
\text{a} \\
\uparrow \\
\text{\mu}
\end{array} \]

For the charge current \( j_\mu = \bar{U}_3 \gamma_\mu U_1 \), we have

\[ A_{\mu} = j_\mu = \frac{i}{q^2} \gamma_\mu \]

Interaction with \( j'_\mu = \bar{U}_4 \gamma_\mu U_2 \) is by means of \( j'_\mu A_{\mu} = j'_\mu \frac{i}{q^2} j_\mu \)

It is especially important to remember that when you use these rules with the propagators \( \frac{1}{p-m} \) or \( \frac{1}{\not{r}} \), it is no longer necessary to draw two graphs which differ by time ordering.

\[ \begin{array}{c}
\text{\mu} \\
\downarrow \\
\text{\mu} \\
\uparrow \\
\text{\mu}
\end{array} \quad \begin{array}{c}
\text{\mu} \\
\downarrow \\
\text{\mu} \\
\uparrow \\
\text{\mu}
\end{array}
\]

It is only necessary to draw one graph. This is always the convention used in applications.

**Applications and Discussion**

It happens that \( \mu \) mesons obey (to the best of our present knowledge) exactly the same equations as electrons, with only the mass changed.

Suppose a stationary electron target, with \( \mu \) particles shot in (in the lab frame). The \( \mu \) and e will scatter into new momentum states because of the interaction of their charges

\[ \begin{array}{c}
\text{\mu} \\
\downarrow \\
\text{e} \\
\uparrow \\
\text{\mu}
\end{array} \quad \begin{array}{c}
\text{\mu} \\
\downarrow \\
\text{e} \\
\uparrow \\
\text{\mu}
\end{array}
\]

\[
\text{Rate} = \sigma \sum_{\mu} \frac{-4\pi e^2}{q^2} \left( \bar{U}_{1\mu} \gamma_\mu U_{2\mu} \right) \left( \bar{U}_{3\mu} \gamma_\mu U_{1\mu} \right) \frac{d^4p_4}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{d^4p_\mu}{(2\pi)^4} \frac{1}{2m_e} \frac{1}{2E_{\mu}} \left( \frac{1}{1 + M^2} \right)^2
\]

The \( \sum \) functions may then be unraveled against the differentials in any order you find convenient (hint: the answer is usually pretty messy no matter what order you use, but there may be fewer messy steps in one sequence than in another).

Because the \( \mu \) and electron have the same interaction with the electromagnetic field, we can write down the matrix element by using the same rules as for electron-
It is easy to see that the largest contributions to the cross section come from \( p_x = p_y \) (forward scattering in the center of mass system). When \( \vec{q} \) is small, the interaction radius (the Fourier transform conjugate of \( \sqrt{\vec{q}} \)) is large. This means most of the cross section is obtained from cases where the particles aren't close together. Hence this is not a good experiment to do if you want to test the laws of electrodynamics - we know already that they work fine for large separations because this is the classical limit. What we want to examine closely is whether the laws break down when the particles get close together. This would be found by looking at large \( \vec{q} \) behavior of the matrix element. But \( \vec{q} \) for this particular experiment is extremely small even for high energy \( \mu \) 's.

Question: if it were possible to have a target of \( \mu \) 's and shoot electrons at them, would it be easier to explore the large \( \vec{q} \) limit?

For education: is it true that to lowest order the scattering of \( \mu^+ e^- \) can be obtained from the scattering of \( \mu^- e^- \) by the replacement \( J_\mu \to -J_\mu \)?

Why?

One of the most precise experiments to date along this line is the scattering

\[ e^- p \to e^- p \]

Here the matrix element is again proportional to \( \frac{J_\mu \bar{u}_\gamma u_\gamma}{(p_1 - p_2)^2} \), where \( J_\mu \) now represents the electromagnetic current of the proton.

It can be shown that the most general form of \( J_\mu \) is

\[ J_\mu = \bar{u}_3 \left( \gamma_\gamma \gamma_\mu \gamma_{\gamma\mu} - \gamma_\gamma \gamma_\mu \right) q \right) \mathcal{U}_1 \]

where \( u_3 \) and \( \mathcal{U}_1 \) are nucleon plane wave spinors and \( F_1 \) and \( F_2 \) are assumed for theoretical reasons to be functions only of the exchanged 4-momentum transfer squared \( (q^2) \)

(Justify this form for \( J_\mu \) by fiddling with others)

It is found empirically that both \( F_1 \) and \( F_2 \) have the shape

\[ \frac{K_{1,2}}{(q^2 - \Lambda^2)^2} \]

where \( \Lambda \) is
is a universal constant. One could, therefore, get the same result just by modifying the photon propagator. It is now considered more acceptable to ascribe the form factors $F_i$ to mesonic substructure of the nucleons, rather than to a breakdown of QED.
Diff. cross section for Compton Scattering in lab ($\beta = 0$):

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi E_2 w_2} \left(\frac{1}{2m}\right)^2 \frac{1}{m^2} \frac{1}{m + 2w_2 \sin^2 \theta/2} \quad \text{d} \omega = \text{spherical solid angle of outgoing photon.}$$

$$w_2 = \text{energy of incoming } \gamma \quad w_2 = \text{energy of outgoing } \gamma \quad E_2 = \text{energy of outgoing } e^-.$$

From kinematics, \( \frac{w_2}{w_1} = \frac{m}{m + 2w_2 \sin^2 \theta/2} \)

$$\frac{d\sigma}{d\Omega}\text{lab} = \frac{1}{64\pi^2} \frac{1}{m} \frac{1}{m^2} \frac{w_2^2}{w_1^2} \quad \ldots \quad (1)$$

This cross section refers to a definite process in which the initial and final states are completely specified defined "pure" states.

Now there are 2 possible helicities for the electron in the initial state, as well as 2 possible helicities for the electron in the final state. In addition, there are 2 possible polarization states for the photon in both the initial and final states. There are, therefore, a total of 16 polarized differential amplitudes \( M \) that occur in Compton scattering.

**Kinematics of scattering in the lab system**

\[ P^\gamma + k^\gamma - P^e - k^e \]

Initial state spinors

\[ u_1 = u_{1+} = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_1 = u_{1-} = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
Final state spins:

\[ U_2 = U_{2+} = \begin{pmatrix} \sqrt{E+m} \cos \phi \\ -\sqrt{E+m} \sin \phi \\ \sqrt{E-m} \cos \phi \\ -\sqrt{E-m} \sin \phi \end{pmatrix}, \quad \text{or} \quad U_{2-} = \begin{pmatrix} \sqrt{E+m} \sin \phi \\ \sqrt{E+m} \cos \phi \\ -\sqrt{E-m} \sin \phi \\ -\sqrt{E-m} \cos \phi \end{pmatrix} \]

\[ U_{2-} = (\sqrt{E+m} \ 6, \sqrt{E+m} \ \Gamma, \ -\sqrt{E-m} \ 6, \ -\sqrt{E-m} \ \Gamma) \]

\[ U_{2+} = (\sqrt{E+m} \ \Gamma, \ -\sqrt{E+m} \ 6, \ -\sqrt{E-m} \ 6, \ \sqrt{E-m} \ \Gamma) \]

where \[ \begin{aligned} \Gamma &= \cos \phi \frac{1}{2} \\
6 &= \sin \phi \frac{1}{2} \end{aligned} \]

There are 2 possibilities for \( \epsilon_1 \), the polarization of the incoming \( \gamma \):

1. Polarized in plane of scattering \( \epsilon_1 = (1, 0, 0, 0) = \epsilon_{1a} \)
2. Polarized perpendicular to plane of scattering \( \epsilon_1 = (0, 1, 0, 0) = \epsilon_{1b} \)

Similarly for \( \epsilon_2 \):

1. In plane of scattering \( \epsilon_2 = (\cos \theta, 0, -\sin \theta, 0) = \epsilon_{2a} \)
2. Perpendicular to plane of scattering \( \epsilon_2 = (0, 1, 0, 0) = \epsilon_{2b} \)

Now we carry out the computation of \( M \) for the 16 different processes:

In each case, \( M \):

\[ \frac{M}{\text{tev}^2} = \frac{U_{2+} \varepsilon_2 (p_1 + k_1 + m) \gamma_1 U_1 + U_{2-} \varepsilon_2 (p_1 - k_1 + m) \gamma_2 U_1}{(p_1 + k_1)^2 - m^2} + \frac{U_{2+} \varepsilon_2 (p_1 - k_2 - m) \gamma_2 U_1}{(p_1 - k_2)^2 - m^2} \]

\[ = \ U_{2+} \left( \frac{p_1 k_1 \varepsilon_1 + \varepsilon_2 k_2 \gamma_2}{2m \omega_1} \right) U_1 \quad \text{since} \quad p_1 \varepsilon_1 = -\varepsilon_2 k_2 \]

\[ k_1 = w_1 (x_1 - y_1) \]

\[ k_2 = w_2 (x_2 - y_2 \cos \theta - x_2 \sin \theta) \]
Case 1: \[ u_1 = u_{1r} \quad u_2 = u_{2r} \]
\[ e_1 = e_{1r} \quad e_2 = e_{2r} \]

Then \[ -y_1 = y_r \quad -y_2 = y_2 c - y_1 S = x c - y S \]

according to the abbreviations:
\[ y_r \rightarrow x \]
\[ y_2 \rightarrow y \]
\[ y_1 \rightarrow z \]
\[ y_0 \rightarrow t \]

Then \[ M_1 \cdot 2m \frac{2}{4\pi e^2} = \bar{u}_2 \left[ (x c - y S) (t - z) x + x (t - z c - x S) (x c - y S) \right] u_1 \]
\[ = \bar{u}_2 \left[ t c - y S - t y z S - x S + t c + t z y S - z c^2 - y S \right] \]
\[ + t y z S - y S^2 \]

where repeated use has been made of the commutation rules for the \( y \)'s:
\[ x t + t x = 0 \]
\[ x^2 = -1 \quad \text{etc.} \]

\[ = \bar{u}_2 \left[ 2 t c - x S - z (1 + c) \right] u_1 \]

To evaluate \[ [2 t c - x S - z (1 + c)] u_1 \], we make use of the table of \( \delta \)-matrices on page 7.

Obtain \[ M_1 \cdot 2m \frac{2}{4\pi e^2} = \bar{u}_2 \begin{pmatrix} 2 c & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} \sqrt{2m} = \sqrt{2m} \left( \sqrt{E+m} \Gamma, -\sqrt{E-m} \sigma, \sqrt{E+m} \sigma \right) \frac{2}{4\pi e^2} \]

\[ M_1 \sqrt{2m} \frac{2}{4\pi e^2} = 2 \sqrt{E+m} \Gamma + \sqrt{E-m} (6 S - i C) - \Gamma \sqrt{E-m} \]

\[ = 2 \sqrt{E+m} \cos \theta \cos \frac{\theta}{2} - \sqrt{E-m} \cos (\theta + \frac{\theta}{2}) - \sqrt{E-m} \cos \frac{\theta}{2} \]

or \[ M_1 = \frac{4\pi e^2}{\sqrt{2m}} \left( 2 \sqrt{E+m} \cos \theta \cos \frac{\theta}{2} - \sqrt{E-m} \cos (\theta + \frac{\theta}{2}) - \sqrt{E-m} \cos \frac{\theta}{2} \right) . \]
Table of Independent $\gamma$-matrices:

We write this out in a particular representation defined by

\[ \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Then \( \gamma_5 = \beta \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_i = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

\[ \gamma_x \gamma_y = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma_z \gamma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \gamma_5 = \gamma_x \gamma_y \gamma_z = +i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma_x \gamma_z = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma_x \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma_z \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma_5 \gamma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_x \gamma_y \gamma_z \]
\[ \gamma_5 \gamma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_x \gamma_y \gamma_z \]
\[ \gamma_5 \gamma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_x \gamma_y \gamma_z \]
\[ \gamma_5 \gamma_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma_x \gamma_y \gamma_z \]

\[ \gamma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
2.  $U_{1+} \rightarrow U_{2+}$
   $E_{1a} \rightarrow E_{2a}$

   $M_2 \overline{\nu} \begin{pmatrix} \frac{2m}{4\pi e} \\ \xi \end{pmatrix} = \overline{\nu} \begin{pmatrix} t \xi g x - x y z - t x y + x y z C + y S \frac{y}{2} \end{pmatrix} \begin{pmatrix} u \\ \frac{m}{2} \end{pmatrix}$

   $= \overline{\nu} \begin{pmatrix} t \xi g x - x y z - t x y + x y z C + y S \frac{y}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{m}{2} \end{pmatrix}$

   $= \overline{\nu} \begin{pmatrix} 0 \\ \frac{m}{2} \end{pmatrix}$

   $= \begin{pmatrix} 0 \\ \frac{m}{2} \end{pmatrix}$

   $2 \overline{\nu} \begin{pmatrix} \frac{m}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{m}{2} \end{pmatrix}$

3.  $U_{1+} \rightarrow U_{2-}$
   $E_{1a} = E_{1a} \rightarrow E_{2} = E_{2a}$

   $M_3 \overline{\nu} \begin{pmatrix} \frac{2m}{4\pi e} \\ \xi \end{pmatrix} = \overline{\nu} \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix} \begin{pmatrix} u \\ \frac{m}{2} \end{pmatrix}$

   $= \overline{\nu} \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix} \begin{pmatrix} u \\ \frac{m}{2} \end{pmatrix}$

   $= \overline{\nu} \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix} \begin{pmatrix} u \\ \frac{m}{2} \end{pmatrix}$

   $M_3 \overline{\nu} \begin{pmatrix} \frac{m}{2} \end{pmatrix} = \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix}$

4.  $U_{1+} \rightarrow U_{2-}$
   $E_{1a} \rightarrow E_{2a}$

   $M_4 \overline{\nu} \begin{pmatrix} \frac{m}{2} \end{pmatrix} = \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix}$

5.  $U_{1} \rightarrow U_{2-}$
   $E_{1a} \rightarrow E_{2a}$

   $M_5 \overline{\nu} \begin{pmatrix} \frac{m}{2} \end{pmatrix} = \begin{pmatrix} 2 t C z - z C x - z C S - x S + t C - z C S \end{pmatrix}$
Notice that $M_5 = M_1$. This can be seen to be a consequence of the invariance of the coupling $J^1 \mu_1$ under parity transformation $P$.

Under $P$, $\frac{J^1}{2} (spin)$, like $L (orbital \text{ ang. mom.})$ is invariant. $P^2$, however, changes sign, so $\frac{J^1}{2} P$ also changes sign and we have the result that the helicity of an electron changes sign under parity.

From the form of the amplitude, it is obvious that reversing the directions of polarization of both photons leaves the lowest order amplitude.

Putting these together, we see that

$$M_1 \left( \frac{u_1^+ \rightarrow u_2^+}{e_1^a \rightarrow e_2^a} \right) = M_5 \left( \frac{u_1^- \rightarrow u_2^-}{e_1^a \rightarrow e_2^a} \right) = M_5 \left( \frac{u_1^- \rightarrow u_2^-}{e_1^a \rightarrow e_2^a} \right).$$

Similarly, $M_6 \left( \frac{u_1^- \rightarrow u_2^-}{e_1^a \rightarrow e_2^a} \right) = M_2 \left( \frac{u_1^+ \rightarrow u_2^+}{e_1^a \rightarrow e_2^a} \right)$

$$M_7 \left( \frac{u_1^- \rightarrow u_2^+}{e_1^a \rightarrow e_2^a} \right) = M_5 \left( \frac{u_1^- \rightarrow u_2^-}{e_1^a \rightarrow e_2^a} \right)$$

$$M_8 \left( \frac{u_1^- \rightarrow u_2^+}{e_1^a \rightarrow e_2^a} \right) = M_4 \left( \frac{u_1^+ \rightarrow u_2^-}{e_1^a \rightarrow e_2^a} \right).$$

9. $\frac{u_1^+ \rightarrow u_2^+}{e_1^b \rightarrow e_2^b}$, $-\phi_1 = \phi_2$

$$M_9 \frac{\sqrt{2m}}{4\pi e^2} = \left( \frac{2VE+m}{E-m} \cos \theta_1 - \frac{2VE-m}{E+m} \cos \theta_2 \right)^2.$$

10. $\frac{u_1^+ \rightarrow u_2^+}{e_1^b \rightarrow e_2^a}$, $-\phi_1 = \phi_2$

$$M_{10} \frac{\sqrt{2m}}{4\pi e^2} = \frac{1}{\sqrt{2m}} \overline{u}_2 (xyz C - y S - xy) u_1 = i \sqrt{E-m} \left( \cos \theta_2 - \cos \left( \theta + \phi_2 \right) \right).$$
11. \( U_{1-} \rightarrow U_{2-} \)
   \( e_{1b} \rightarrow e_{2b} \)  
   By parity, \( M_{11} = M_{11} \)

12. \( U_{1-} \rightarrow U_{2-} \)
   \( e_{1b} \rightarrow e_{2a} \)  
   By parity, \( M_{12} = M_{10} \)

13. \( U_{1+} \rightarrow U_{2-} \)
   \( e_{1b} \rightarrow e_{2b} \)

   \( M_{13} = M_{14} \)

\[
\frac{\sqrt{2} m}{4 \pi e^2} M_{13} = 2 \frac{U E + m \sin \frac{\theta}{2}}{U E - m \sin \frac{\theta}{2}} + \frac{U E - m \sin (\theta + \frac{\pi}{2})}{U E - m \sin \frac{\theta}{2}} \]

15. \( U_{1+} \rightarrow U_{2-} \)
   \( e_{1b} \rightarrow e_{2a} \)

   \( M_{15} = M_{16} \)

\[
\frac{\sqrt{2} m}{4 \pi e^2} M_{15} = \frac{1}{\sqrt{2} m} \left( x y z e^{-y z} \right) U_{1+} \quad \text{(Same form as 10.)} \]

\[
= i \frac{U E - m}{U E - m \cos (\theta + \frac{\pi}{2})} \]

\[
= i \left( U E - m \sin (\theta + \frac{\pi}{2}) - U E - m \sin \frac{\theta}{2} \right) \]

The various polarized cross sections are obtained by the replacement \( M \rightarrow M_k \), \( k = 1, 2, \ldots, 16 \) in Equation (1).
Pair production in matter

A photon may create an $e^+e^-$ pair in the presence of a proton according to the following diagram:

\[
\begin{array}{c}
\text{P}_1 \\
\text{P}_2 \\
\text{P}_3 \\
\text{P}_4 \\
\text{proton}
\end{array}
\]

\[-\text{p}_1 = \text{4 momentum of } e^+ \\
\text{p}_2 = \text{4 momentum of } e^-
\]

The piece of the matrix element that is represented by this diagram is

\[
(N \hbar c^2)^3 \frac{J_\mu}{q^2} \overline{u}_2 \gamma \mu \frac{1}{p_1 + q} \gamma \mu \gamma \tau U_1
\]

where $J_\mu$ = matrix element of the electromagnetic current operator taken between the initial and final free proton states.

From general arguments of relativity and charge and parity conservation, it can be shown that if spinor solutions of the free particle Dirac equation are used to describe the free proton, then

\[
J_\mu = e \overline{u}_4 \left\{ \gamma_\mu F_1(q^2) + \frac{1}{4M} q_\mu \left[ \gamma_\nu, \gamma_\mu \right] F_2(q^2) \right\} ^2 U_3
\]

\[
= e \overline{u}_4 \left\{ \gamma_\mu F_1 + \frac{1}{4M} [q_\mu, \gamma_\nu] F_2 \right\} ^2 U_3
\]

The threshold behavior of $F_1$ and $F_2$, that is $F_1(0)$, $F_2(0)$, may be determined in terms of the charge and magnetic moment of the proton by considering the
non-relativistic limit of the coupling of this interaction to an external static field \( A_\mu = (\phi, \vec{A}) \), and identifying terms in the classical formula
\[
H^{NR}_I = e\phi + \vec{\mu} \cdot \vec{B}
\]
In this manner, one obtains
\[
F_1(0) = 1 \\
F_2(0) = \frac{e}{2M} (F_1(0) + F_2(0))
\]
\( F_2(0) \) is the anomalous magnetic moment of the proton, in nuclear magnetons.

In lowest order there is another diagram that must be considered for the process: \( \gamma + p \rightarrow \gamma + p + e^+ + e^- \), and that is

If we neglect the recoil of the nucleus—that is, if we treat the nucleus as an external, static Coulomb field, we have only two diagrams in lowest order:

\[ p_2 = p_1 + k + q \]
\[ q = p_2 - p_1 - k \]
The amplitudes associated with these diagrams, $M_I$ and $M_\Pi$, are

$$M_I = \overline{u}_2 \gamma^\mu(q) \frac{1}{p^2 + \not{k} - m} \gamma^\nu u_1 \sqrt{4 \pi e^2} \rho_{\mu \nu}$$

and

$$M_\Pi = \overline{u}_2 \gamma^\mu(q) \frac{1}{p^2 - \not{k} - m} \gamma^\nu u_1 \sqrt{4 \pi e^2} \rho_{\mu \nu}$$

where

$$a_\mu(q) = \int A_\mu(x) e^{i q \cdot x} dx$$

$$A_\mu = \left( \frac{Ze^2}{R}, 0 \right)$$

$$\alpha'(q) = Y_t \frac{Z e^2}{R} \delta(q_t) e^{-i \vec{q} \cdot \vec{R}}$$

$$\equiv 2 \pi \delta(q_t) \frac{Ze^2}{2 \pi} \frac{\delta_t}{\vec{q}^2 + \frac{1}{2} m^2}$$

If we took into consideration the effect of electron shielding of the field of the nucleus by $V(R) = \frac{Ze^2}{R} e^{-br}$, we'd obtain

$$\alpha'(q) = 2 \pi \delta(q_t) \frac{4 \pi e^2 Z \delta_t}{\vec{q}^2 + \frac{1}{2} m^2}$$

Substituting $\alpha'$ into $M_I + M_\Pi$, and squaring, and multiplying by the appropriate kinematical factors we obtain the cross-section for pair production in the field of a nucleus. The result is an ugly mess which is discussed in Heitler, Quantum Theory of Radiation, § 26. One simple result is that if the e- e+ pair are produced with relativistic energies, they are created primarily in the forward direction within the cone defined by the angle $\Theta = \frac{m}{E}$. For smaller energies, the effect is less marked.
Another phenomenon of considerable interest is the emission of a photon by an electron in the field of a nucleus ("Bremsstrahlung").

The matrix element for this process has the same form as that for pair production by a photon in the field of a nucleus.

\( \delta \)'s create pairs which emit \( \delta \)'s by Bremsstrahlung, which in turn create pairs, ... thus creating a "shower."

The "common sense" infinity of quantum electrodynamics.

It is not possible for scattering to take place without the emission of photons. That is because the field far away from the charge must change if the particle is deflected. It can change only if photons are emitted.

There is some interest, therefore, in an approximate formula for Bremsstrahlung with the emission of low-frequency photons.

Consider the emission of one photon. The relevant diagrams are

\[
 Rate = 2\pi \int \frac{d^3p_2}{(2\pi)^3 E_2 E_k} \frac{d^3k}{2\hbar (2\pi)^3} |M|^2
\]

\[
 M = \left\{ \bar{u}_2 \frac{1}{p_{2+k}-m} \bar{u}_1 + \bar{u}_2 \frac{1}{p_{1-k}-m} \bar{u}_1 \right\} \sqrt{\frac{4\pi \hbar^2}{}}
\]
we are interested only in low-energy photons, so the two terms with arrows above them are small compared to the other terms in the expression and can be neglected to an accuracy of the order \( \frac{k}{m} \).

Then

\[
M = \frac{\bar{u}_2}{-2p_2 \cdot k} \left( \frac{\epsilon_1 + 2p_2 \cdot e + \epsilon_2}{2p_2 \cdot k} \right) u_1 u_{1\prime} + \frac{\bar{u}_2}{-2p_1 \cdot k} \left( \frac{\epsilon_1 - 2p_1 \cdot e}{2p_1 \cdot k} \right) u_1 u_{1\prime}
\]

(The \( X \) 's indicate terms that cancel)

\[
= \frac{\bar{u}_2}{2p_2 \cdot k} u_1 u_{1\prime} \left( \frac{p_2 \cdot e - p_1 \cdot e}{p_2 \cdot k - p_1 \cdot k} \right) \sqrt{4\pi e^2}
\]

Rate

\[
= 2\pi \delta(E_f - E_i) \frac{\alpha^3 p_2^2}{(2\pi)^3 2E_2 E_i} u_1 u_{1\prime}^2 \left( \frac{\bar{u}_2}{2p_2 \cdot k} \right)^2 \frac{1}{2} \left( \frac{p_2 \cdot e - p_1 \cdot e}{p_2 \cdot k - p_1 \cdot k} \right)^2 4\pi e^2 \left( \frac{\omega}{\omega} \right) \frac{d\mathbf{S}_{2\prime}}{\omega} \frac{d\mathbf{S}_2}{\omega}
\]

A is the probability of scattering with no photons emitted, in lowest order.

to the approximation we've made \( \gamma = p_2 - p_1 \).

The term \( \frac{3}{\gamma} \) may be simplified

\[
\frac{3}{\gamma} = \frac{v_{1\prime} \cdot e}{1 - v_{1\prime} \cos \Theta_1} - \frac{v_2 \cdot e}{1 - v_2 \cos \Theta_2} = \frac{v_1 \sin \Theta_1}{1 - v_1 \cos \Theta_1} - \frac{v_2 \sin \Theta_2}{1 - v_2 \cos \Theta_2}
\]

= 0 if \( \mathbf{e} \) is perpendicular to the plane of scattering.

\( \varepsilon \): photons polarized in the plane of the collision.
Energy/unit time emitted in range $dx$

$$\text{= Same formula (without the factor of } \frac{1}{\hbar})$$

which is the same as the classical expression.

**Problem:** Two soft photon emission $(k, l)$

Show that rate $= \text{Probability of scattering with emission of no photons}$

$\times \text{probability of emitting one photon (k)}$

$\times \text{probability of emitting other photon (l)}$

There is also a factor of $\frac{1}{2}$ from statistics.

(i.e., show that the two photons are emitted with statistcally independent probabilities)

**Classical case:** A classical charged particle moves with uniform velocity and is suddenly deflected.

What is the radiation like?

Let the trajectory in space-time be represented parametrically by the functions $Z_\mu(x)$

the current density

$$j_\mu(x) = e \int \delta^4(x - Z(x)) \cdot \dot{Z}_\mu(x) \, dx$$

vector potential satisfies the equation

$$\Box A_\mu(x) = j_\mu(x)$$

Take the Fourier transform of both sides

$$k^2 \hat{j}_\mu(k) = \int e^{-ikx} \, j_\mu(x) \, dx$$

$$= \int e^{-ikx} \, \int \delta^4(x - Z(x)) \, \dot{Z}_\mu(x) \, dx \, dx$$

$$= \int e^{-ik \cdot \Pi(x)} \, \dot{Z}_\mu(x) \, dx = \hat{j}_\mu(k)$$
Let the deflection take place at \( z = 0 \), then

\[
0 < z \quad Z_\mu(z) = p \mu \cdot z \\
\lambda > 0 \quad Z_\mu(z) = (p \mu \cdot \lambda)
\]

Then \( g(p) = \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \left( \frac{\Pi_{\mu} \cdot k - \Pi_{\mu} \cdot k}{\Pi_{\mu} \cdot k} \right) \), so \( \omega(p) = \frac{\omega}{k^2} \left( \frac{\Pi_{\mu} \cdot k - \Pi_{\mu} \cdot k}{\Pi_{\mu} \cdot k} \right) \).

This gives the classical expected energy in \( dw = e^2 C(\omega) \omega \) dw

\[
\text{mean energy liberated in } dw = \text{mean energy liberated in } dw \\
\text{quantum-mechanically}
\]

\[
\text{mean no. of photons emitted} = e^2 C(\omega) \omega \frac{dw}{\omega}
\]

**Emission of many soft photons:**

Rate for one photon emission = \( A \left[ k \left( \frac{V_1 \sin B_1 - V_2 \sin B_2}{1 - V_1 \cos B_1 V_2 \cos B_2} \right) \right]^2 \frac{d\omega}{\omega} \frac{d\omega}{\omega} \frac{4\pi e^2}{(2\pi)^2} \omega
\]

Now to the same order in \( e^2 \), the cross-section for emitting no photons is

\[
6 = \sigma_{A}(C+1)
\]

where \( C \) is a correction factor that comes from interference between the two diagrams

\[
\text{which is } O(\omega^2)
\]

\[
\text{i.e., } C = O(e^2)
\]

\[
\text{6 for 1 photon emission} = \sigma_{A} \times (1+C), \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right)
\]

\[
\text{6 for 2 photon emission} = \sigma_{A} \times (1+C) \left[ \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right) \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right) \right]
\]

\[
\text{6 for 3 photon emission} = \sigma_{A} \left( 1+C \right) \frac{1}{3!} \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right) \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right) \left( \frac{d\omega}{\omega} f(\omega) \frac{d\lambda}{\lambda} \right)
\]

etc.
\[ \text{6 scattering with any } \omega, \phi, \Delta \text{'s emitted} \]

The divergence ("infrared catastrophe") comes from the the fact that the integral in the exponential doesn't converge.

However \( C' = e^{-\int \frac{d\omega}{\Delta} f(\omega) d\Omega} \) is the contribution of the low energy virtual photons, so \( \text{6 scattering} \) is finite.

The analysis implies that \( C' \text{ photons} = 6 A C' = 0 \).

Experimentally, there is an energy \( \Delta \) such that photons with a total energy \( < \Delta \) are not observed. So we measure 6 of emitting no photon with energy \( > \Delta \).

\[
\tilde{g}(\Delta, \omega) = 6 \text{ photons} + 6 \text{ photons} + 6 \text{ photons} + \ldots \\
\begin{align*}
&= 6 A C' (1 + \int_{\Delta} \frac{d\omega}{\Delta} f(\omega) + \frac{1}{2} \int_{\Delta} \frac{d\omega}{\Delta} \left( \frac{d\omega}{\Delta} f(\omega) \right)^2 + \cdots ) \\
&= 6 A e^{-\int \frac{d\omega}{\Delta} f(\omega)} \left( 1 + \frac{2 \Delta \log \Delta}{
\frac{\Delta}{\epsilon} + \frac{1}{2} \left( \frac{\Delta}{\epsilon} \right)^2 (\log \frac{\Delta}{\epsilon})^2 + \cdots \right) \\
&= 6 A e^{-\frac{2 \Delta}{\epsilon} (\log \frac{\Delta}{\epsilon} - \log \frac{\epsilon}{\Delta})} \\
&= 6 A e^{-\frac{2 \Delta}{\epsilon} (\log \frac{\Delta}{\epsilon})} = 6A \left( \frac{m}{\Delta} \right)^{-\frac{2\Delta}{\epsilon}} \\
\end{align*}
\]

we will show later how to calculate \( C' \).
Additional rules for calculating diagrams:  

\[ M_A = (\frac{1}{2} \pi e^2)^2 \frac{\bar{u}_2 \gamma_\nu u_1 \bar{u}_3 \gamma_\nu u_4}{(p_1 - p_3)^2} = A \]

\[ M_B = -(\frac{1}{2} \pi e^2)^2 \frac{\bar{u}_4 \gamma_\nu u_3 \bar{u}_1 \gamma_\nu u_2}{(p_1 - p_3)^2} \]

A is the full amplitude for e⁻-μ⁻ scattering but not for e⁻-e⁻ scattering because there is the exchange possibility:

The minus sign in this contribution to the amplitude comes from the rule of Fermi statistics for electrons, because B differs from A only in that the two outgoing electrons are interchanged.

\[ \theta^- - \theta^+ \text{ scattering} \]

\[ M_B = -p_4 = \text{momentum of incoming } e^+ \]

The matrix element is the same as A above.

What corresponds to the exchange diagram for e⁻-e⁻ scattering?

The analogy is the annihilation diagram:

This is the same as the first diagram, rotated 90° together with the exchange of two electron exit lines (3 and 4) \( \Rightarrow \text{relative minus sign in the amplitude.} \)
That this additional term in the $e^-e^+$ amplitude is present has been checked experimentally to better than 10% by observations of the energy level spectrum of positronium. The "annihilation force" causes a shift in the energy levels from those expected on the basis of pure coulomb attraction.

A technique for calculating unpolarized cross sections:

\[ \text{Rate} = \sum_{\text{spins}} |\overline{u}_2 N u_1|^2 \]

Suppose the incoming particle is unpolarized and the spin of the final particle is not measured.

Then rate = \( \sum_{\text{spins}} |\overline{u}_2 N u_1|^2 = \sum_{\text{spins}} \overline{u}_2 N u_1 \cdot (\overline{u}_2 N u_1)^* \)

Let \( N \) be defined by \( (\mathbf{g} \cdot N f)^* = (\mathbf{f} \cdot N g) \) (it is easy to see \( \mathbf{N} = \mathbf{y}_1 \mathbf{N} \mathbf{y}_2 \))

Then rate = \( \sum_{\text{spins}} |\overline{u}_2 N u_1|^2 \)

\[ X = \sum_{\text{spins}} (\overline{u}_2 N u_1) (\overline{u}_2 N u_1) = \sum_{\text{spins}} \overline{u}_2 N \frac{p_1^\gamma + m}{2m} u_1, \overline{u}_1 N u_2 \]

\[ = \sum_{\text{spins}} \overline{u}_2 N \frac{p_1^\gamma + m}{2m} u_1, \overline{u}_1 N u_2 \]

\[ \text{since } (p_1^\gamma + m) u_3 = 0; (p_1^\gamma + m) u_4 \]

Now \( \overline{u}_1 u_1 = 2m I \), \( I = \text{identity matrix} \)

So \( X = \sum_{\text{spins}} \overline{u}_2 N \frac{p_1^\gamma + m}{2m} N u_2 = \sum_{\text{spins}} \overline{u}_2 N \frac{p_1^\gamma + m}{2m} \frac{p_2^\gamma + m}{2m} u_1 \)

\[ = \text{trace} \left( N (p_1^\gamma + m) (p_2^\gamma + m) \right) \]

Aids: \( \text{Tr} I = 4, \text{Tr} \gamma_\alpha = 0, \text{Tr} \gamma_\alpha \gamma_\beta = 0, \text{Tr} (p_1^\gamma p_2^\gamma) = 0, \text{Tr} (p_1^\gamma) = 0. \)
INTENSITY DISTRIBUTION OF BREMSSTRAHLUNG RADIATION

Consider the factor \( \frac{\beta_1 \cdot e}{\beta_1 \cdot k} - \frac{\beta_2 \cdot e}{\beta_2 \cdot k} \), which may be interpreted as the probability that an electron emit a soft photon of momentum \( k \) in going from momentum \( \beta_1 \) to momentum \( \beta_2 \).

\[
\beta_1 \cdot k = \hbar \left[ E_1 - \beta_1 \cdot e \cdot \Omega_1 \right]
\]

\[
\beta_1 \cdot e = \beta_1 \cdot e_2 - \beta_1 \cdot e_1 = \beta_1 \cdot e \cdot \Omega
\]

Hence we obtain a function of shape
\[
\frac{\sqrt{1 - \beta_1^2 \cdot \Omega_1^2}}{1 - \beta_2^2 \cdot \Omega_2^2}
\]

A non-relativistic interpretation of this is to examine the electron before and after

The electron is accelerated in the direction \( \Delta \vec{r} \) by the scattering, and hence must emit light. Detailed comparison of the above result with classical results (see Jackson, around p. 472) is left as a exercise for the student.

In the highly relativistic region, where \( \beta_1 \approx c \),
\[
\beta = \sqrt{E^2 - m^2} \rightarrow E - \frac{m^2}{2E}
\]

Thus
\[
\frac{\beta \cdot \sin \Theta}{E - \beta \cdot \omega \cdot \Theta} \rightarrow \frac{E \cdot \Theta}{E^2 \left[ \Theta^2 - \frac{m^2}{E^2} \right]} \sim \frac{\Theta}{\Theta^2 + \left( \frac{m}{E} \right)^2}
\]

This takes on the shape
\[
\frac{\Theta}{m/c} = \frac{\Theta}{\gamma}
\]

When the deflection angle due to scattering is big compared to \( \frac{\Theta}{\gamma} \), then either one or the other of the above terms is big, and the radiation emitted takes on the shape plotted above

For small angle scattering the shape gets more complicated

Notice that if you emit two identical particles, some care must be taken in obtaining a final answer.

Suppose you wish to measure the differential cross section and total rate for a process in which 2 photons are emitted, along with some other stuff. Assume you have one photon counter.
Every time any photon hits the counter, it records a count.

For each such count, the other photon could have gone anywhere. Thus the rate measured by the counter when it is placed at angle $\theta_1$ is $\frac{d\sigma}{d\theta_1}$ - the differential cross section for particle 1, where we define particle 1 as the one that hit the counter.

That is, the answer got by integrating the familiar expression over $\theta_1$ gives the cross section to measure the emission of some photon into angle $d\theta_1$. We have no way of telling whether this is the "red" photon or the "blue" one in the Feynman diagram, and we don't care.

If we now integrate over angles $\theta_1$, we are summing cases like

$$\begin{array}{c}
\theta_1 \\
\theta_2 \\
\theta_3
\end{array}$$

where 1 and 2 have been assigned simply on the basis of the counter position.

These cases are indistinguishable.

To get a total rate, all we want is the probability that the reaction went.

$$\int_0^{2\pi} \int_0^{2\pi} \frac{d\sigma}{d\theta_1 d\theta_2}$$

gives twice this probability. Hence, in obtaining a total rate, we must divide the final integral by $n!$, where $n$ is the number of indistinguishable particles in the final state.

MORE ABOUT THE DIRAC EQUATION

Velocity in Dirac language

$$\hat{H} = \gamma \cdot \hat{\mathbf{v}} + \frac{1}{2} \cdot (\hat{\mathbf{p}} - \hat{\mathbf{A}}) \cdot \hat{\mathbf{v}} \Rightarrow \dot{\mathbf{x}} = i \left[ \hat{H}, \mathbf{x} \right] = \alpha \mathbf{x}$$

Hence one would like to find some interpretation for $\alpha$ in terms of particle velocity

However, $\alpha^2 = 1$; thus the eigenvalues are $\pm 1$, and this seems to imply that a measurement of the velocity in the $x$ direction would yield $\pm C$.

This has caused some concern.

Dirac's explanation:

To measure the velocity, you must measure the position twice. But the first accurate measurement of position would make the momentum totally uncertain, and thus you would measure $0$. 
A measurement of average velocity, given by 
\[ \mathbf{v}^2 = \left( \frac{x_{t+T} - x_t}{T} \right)^2 \]
would give roughly \( \left( \frac{p}{\sqrt{\gamma^2 + m^2}} \right)^2 \) if \( T > \frac{\hbar}{m} \)
(can you show this?) but measurement of the instantaneous value would come out wrong.

Feynman’s explanation

Dirac’s logic is wrong. For \( [\alpha, \beta] = 0 \) implies they can be measured simultaneously. If one fumbles around, one finds that it is indeed possible to construct a solution of the Dirac equation which has a definite eigenvalue of \( \alpha \) and \( \beta \); but such a solution does not have a definite energy. In order to be a solution with definite momentum \( p \), it must have energy \( \sqrt{\gamma^2 + m^2} \). Hence the solution in question must be some mixture of electron and positron.

Given that the system must have a certain net charge, the Dirac equation can describe systems with one electron or one electron and pair(s); thus \( \alpha \) is not necessarily a single particle operator and might better be thought of as a sort of current density.

The commutators of such operators as \( \alpha \beta \rho \gamma \), \( \beta \alpha \beta \gamma \rho \) etc. with the Hamiltonian have not been completely explored.

RELATIVISTIC INVARIANCE OF DIRAC EQUATION

So far we have calculated all answers with a given set of \( \gamma \) matrices. But we have assigned a Lorentz index to these matrices and treated them like a 4 vector. How, then, do we know we are using the right \( \gamma \)'s? Why don't we use some \( \gamma' = \gamma' \gamma_0 \) with the transformation \( \delta \gamma' \) depending on the frame of the problem?

A clever answer to this is to say that if you read your Dirac equation off some moving system with a telescope, it wouldn't change anything. This argument, however, doesn’t get to the core of the problem. The explanation of this apparent paradox is that \( \gamma_\mu \) and \( q_\mu \), \( \gamma_\nu \) are related by an equivalence transformation; provided they are used with solutions of the Dirac equation that are transformed in the same way, all answers will be independent of the representation used.

To discover in general the conditions that we have the same physics with different matrices
\[
\left[ \gamma_\mu \left( i \gamma_5 - A_\mu \right) + m \right] \gamma^\nu = 0
\]

Define \( \gamma'' = S \gamma \), where \( S \) is a matrix of constants

Then \( \left[ \gamma''_\mu \left( i \gamma_5 - A_\mu \right) + m \right] \gamma''^\nu = 0 \) will be equivalent to the above if \( S \gamma_\mu S^{-1} = \gamma''_\mu \), \( \gamma''_\mu = S^{-1} \gamma''^\nu S \)

This sort of transformation preserves all algebraic relations between the matrices (in particular their commutation relations).

However, we must demand one more thing in order that the physics be unchanged.

Matrix elements must also be preserved, \( \overline{\gamma} A \gamma = \overline{\gamma''} A'' \gamma'' \).

Hence \( \overline{S} = \gamma_\nu S^+ \gamma_\nu \) must equal \( S^{-1} \).

Then an equivalence transformation by any \( S \) such that \( \overline{S} = S^{-1} \) leaves everything unchanged.

**Relativistic Adjoints**

\[
(\overline{F M q})'' = \overline{S} \overline{M} \frac{d}{d}\overline{q}
\]

defines \( \overline{M} = \beta M + \beta \)

For matrices \( A, B, C \) and constants \( \lambda \),

\[
\left( \lambda A B \ldots C \right) = \lambda^m A^m \ldots B B
\]

\( \overline{\gamma_M} = \gamma_M \)

Hence the relativistic adjoint of any number of \( \gamma \) matrices is got simply by reversing their order.

**EXAMPLES of Useful Equivalence Transformations**

1) Define \( \gamma \) by \( \overline{\gamma_\nu} = \gamma_{\nu m h} \gamma \)

Then the velocity transformations assume a form similar to that for rotations

For the velocity transform in a direction,

\[
\gamma_\nu \rightarrow (\cosh \omega) \gamma_\nu - \gamma_\nu \sinh \omega
\]

\( \gamma_\nu \rightarrow \gamma_\nu \)

\( \gamma_\gamma \rightarrow \gamma_\gamma \)

\( \gamma_\nu \rightarrow (\cosh \omega) \gamma_\nu - (\sinh \omega) \gamma_\nu \)

If \( S = e^{\omega_\gamma \gamma_\gamma} \gamma_\gamma \), then \( S^{-1} = e^{-\omega_\gamma \gamma_\gamma} \gamma_\gamma \gamma_\gamma = \overline{S} \)

\[
\overline{S} = \gamma_\gamma \gamma_\gamma \gamma_\gamma = \gamma_\gamma e^{\omega_\gamma \gamma_\gamma} \gamma_\gamma \gamma_\gamma = e^{-\omega_\gamma \gamma_\gamma} \gamma_\gamma \gamma_\gamma \gamma_\gamma = e^{-\omega_\gamma \gamma_\gamma} \gamma_\gamma \gamma_\gamma \gamma_\gamma
\]

\( \gamma^2 + \gamma \gamma = 1 + A + \frac{1}{2} A A + \frac{1}{3} A A A + \ldots \ldots \)

Expansion shows that

\[
S = \cosh \omega \gamma_\gamma \gamma_\gamma + \gamma_\nu \gamma_\gamma \gamma_\gamma \sinh \omega \gamma_\gamma \gamma_\gamma
\]

\( S^{-1} = \cosh \omega \gamma_\gamma \gamma_\gamma - \gamma_\nu \gamma_\gamma \gamma_\gamma \sinh \omega \gamma_\gamma \gamma_\gamma \)

2) Show that the corresponding rotation operator is \( S = e^{\frac{1}{2} \gamma_\gamma} \gamma_\gamma \gamma_\gamma \gamma_\gamma \)

Then
\[\psi'(x, y, z, t) = e^{i \int_{\mathbf{J}_z} \mathbf{\Theta} \cdot \mathbf{A}} \psi(x, y, z, t) - \mathbf{A}(x, y, z, t) \cdot \mathbf{\Theta}\]

(work through the algebra to get some familiarity)

Using this definition of \(\mathbf{J}_z\), one can show for the Dirac equation that if there is no vector potential, and if \(\nabla(\mathbf{v}) = \nabla(\mathbf{u})\), then angular momentum is a constant of the motion.

**PROBLEM:** See how much you can get from the non-relativistic point of view of the formula

\[2m \Delta m = (\text{const}) e^2 y_4 \pi \int \frac{\mathbf{u} \cdot \mathbf{u} (\mathbf{\rho - k} + m) \mathbf{u} \cdot \mathbf{u}}{\{(\mathbf{\rho - k})^2 - m^2\}^2} d^4 k\]

for the correction in mass of an electron of momentum \(p\) due to second order interactions with the electromagnetic field.

**Hint:** If you start with transverse waves you will get only the \(\gamma_\perp\) piece. There is also a longitudinal contribution to the self energy, which looks like \(e^2/r_{\perp}\).

**COMPUTATIONAL AIDS**

Spin summations for electrons and positrons

\[\sum_{\text{spins}} \frac{1}{u_2} A u_1 \overline{u}_1 B u_2 = \sum_{\text{all } u_1} \frac{1}{u_2} A (u_1^* + m) u_1 \overline{u}_1 B u_2 = \frac{1}{u_3} A (p_3 + m) \overline{u}_3 B u_2\]

Thus

\[\sum_{u_1 \text{ spins}} \left| \frac{1}{u_2} M u_1 \right|^2 = \frac{1}{u_2} M (p_3 + m) \overline{M} u_2\]

\[\sum_{u_2 \text{ spins}} \frac{1}{u_2} M (p_3 + m) \overline{M} u_2 = Tr \left[ M (p_3 + m) \overline{M} (p_3 + m) \right]\]

To calculate the traces:

\[T_{\lambda} \lambda = 4, \quad Tr \gamma_\perp = 0\]

Trace of any odd number of \(\gamma_\perp\) is 0.

**Terminology**

\[Tr (X) = S_{\rho, \mu, \nu} (X)\]

\[Tr (AB) = T_{\lambda} (B A) = T_{\lambda} (A B)\]

\[Sp \rho = 4 \rho_{\perp} \quad Sp \rho = 0 = Sp (\rho_\perp \rho_\perp)\]

\[Sp \rho \rho = 4 \left[ a \cdot b \cdot c \cdot d - a \cdot b \cdot c \cdot d + a \cdot b \cdot c \right]\]

Always try to use the Dirac equation \(\gamma \cdot u = m u\)

**to reduce the number of \(\gamma\) matrices in the matrix element before taking traces**

There are tricks for taking traces of any number of \(\gamma\) matrices, but in practice even 6 \(\gamma\) matrices in a row lead to headaches. If you find you have to take traces of more than 4 \(\gamma\)s, it is worth trying to find some simplification. One helpful thing in problems involving photons is
Suppose you get something like

$$\sum_{\text{light polarizations}} S_{\rho} \left[ \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right]$$

Provided you have not used special properties of \( \xi \), in some gauge (like assuming it has no time component) then the summation over transverse directions of the light can be replaced by

$$\xi_1 \rightarrow \gamma_\mu \rightarrow \sigma_\mu$$

where summation is implied (also multiply the mess by -1)

Reason:

Gauge invariance implies that replacing \( \xi \) by \( h \) should give 0. If \( \xi \leftrightarrow (\nu, e, \gamma, \nu) \) then \( h \leftrightarrow (k, \nu, \nu, k) \)

and gauge invariance says

$$h \left[ \left( \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right) - \left( \begin{array}{c} \xi_2 \\ \xi_2 \end{array} \right) \right] = 0$$

Hence \( \gamma_\mu \rightarrow \gamma_\mu \) boils down to \(- \left( \begin{array}{c} \gamma_x \\ \gamma_x \end{array} \right) + \left( \begin{array}{c} \gamma_y \\ \gamma_y \end{array} \right) \)\)

which is what you expect from the property \( \gamma_\mu \gamma^\nu = \delta_\mu^\nu \) (1 and \( \nu \) are spacelike)

Once the quantity is in the form

$$\gamma_\mu \gamma_\mu$$

one then uses

$$\gamma_\mu \gamma_\mu = 4$$
$$\gamma_\mu \gamma_\nu = -2 \delta_\mu^\nu$$
$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 4 \delta_\mu^\nu \delta_\rho^\sigma$$

Thus each summation over photon polarizations reduces the number of \( \gamma \) matrices in the trace by 2

**SUMMARY OF RULES**

$$\text{Prob of Transition sec}^{-1} = \frac{1}{2m} p(E) \left\langle \right| \frac{1}{2} \frac{1}{2} \left( \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right) \right| m \right\rangle^2$$

$$= (2\pi)^4 \int (\hat{\rho}_m - \hat{\rho}_0) \left\langle \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right| \frac{1}{2} \frac{1}{2} \left( \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right) \right| m \right\rangle^2$$

where \( m \) is a relativistically invariant matrix

Using the normalization \( \overline{m} m = 2m \)

\( \left\langle \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right| m \right\rangle \neq 0 \)

gives as the sum over states

$$\sum_{\rho} \left\langle \begin{array}{c} \xi_f \nu + m \end{array} \right| m \right\rangle$$

Calculation of
Electron propagator \( \frac{i}{p^2 - m^2} \epsilon \)

Photon propagator \( \frac{-i}{\hbar^2 c^2 \epsilon} \)

Spin 0 meson propagator \( \frac{i}{p^2 - m^2 + i\epsilon} \)

Couplings:

Fermions to a real photon \( -i \not{\epsilon} \sqrt{4\pi \hbar^2} \)

Fermions to a potential \( \Psi(\mathbf{q}) = \int e^{-i \mathbf{q} \cdot \mathbf{x}} A(\mathbf{x}) \delta^3 \mathbf{x} \)

To a virtual photon \( -i \sqrt{4\pi \hbar^2} \gamma_\mu \cdots - \gamma_\mu \sqrt{4\pi \hbar^2} \)

To pseudoscalar meson \( \sqrt{4\pi \hbar^2} \gamma_5 \cdots - \gamma_5 \sqrt{4\pi \hbar^2} \)

To scalar meson \( \sqrt{4\pi \hbar^2} \cdots \sqrt{4\pi \hbar^2} \)

For a closed loop of electrons, the rule is \( -S \rho \).

If there is an indeterminate momentum, sum by

\[
\int \frac{d^4 p}{(2\pi)^4}
\]
The correction to the energy of a free electron due to the interaction of the electron with its own electromagnetic field is represented in lowest order perturbation theory by the diagram:

\[
\Delta E^2 = 4\pi e^2 \int \frac{\delta \mu}{p-k} \frac{\delta \mu}{\vec{u}} \frac{d^4 k}{p-k-m} \frac{d^4 k}{(2\pi)^4 k^2}
\]

and this corresponds to

\[
\Delta E^2 = \frac{4\pi e^2}{(2\pi)^4} \int \frac{\delta \mu}{p-k} \frac{\delta \mu}{\vec{u}} \frac{(p-k+m)^2}{(p-k)^2-m^2+i\epsilon} \frac{d^4 k}{(p-k)^2-m^2+i\epsilon}
\]

This formula may be obtained from the expression for the second order correction to the energy of a state in ordinary perturbation theory:

\[
\Delta E_i = \sum_n \frac{\langle H' \rvert H \rangle_i}{E_i - E_n}
\]

with \( H' \) perturbation

\( = e(\vec{p} \cdot \vec{A} - \phi) \) in our case

\( \Delta E^2 \) will turn out to be logarithmically divergent.

\[
\Delta E^2 = \frac{4\pi e^2}{(2\pi)^4} \int \frac{\delta \mu}{p-k} \frac{\delta \mu}{\vec{u}} \frac{(-2\vec{p} + 2\vec{k} + 4m)}{\vec{k}^2+i\epsilon} \frac{d^4 k}{(p-k)^2-m^2+i\epsilon}
\]

\[
= \frac{4\pi e^2}{(2\pi)^4} \int \frac{(-4m^2 + 4p.k + 8m^2)}{(4k^2+i\epsilon) [(p-k)^2-m^2+i\epsilon]} \frac{d^4 k}{(4k^2+i\epsilon) [(p-k)^2-m^2+i\epsilon]}
\]

since \( u \delta \mu = \bar{u} u \delta \mu \).
For the correction to the rest energy, put $p^2 = 0$

Then $p \cdot k = \omega m$

\[(p - k)^2 = k^2 + m^2 - 2\omega m\]

and

\[\Delta m^2 = \frac{4\pi e^2}{(2\pi)^4} \int d^4k \frac{m(m + \omega)}{(\omega^2 - K^2 + i\varepsilon)(\omega^2 - K^2 - 2\omega m + i\varepsilon)} \]

\[= \frac{e^2}{16\pi^3} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} 4\pi K^2 dK \frac{m(m + \omega)}{(\omega^2 - K^2 + i\varepsilon)(\omega^2 - K^2 - 2\omega m + i\varepsilon)} \]

The integrand has poles as a function of $w$ at

\[w = K - i\varepsilon = w_1\]
\[w = -\sqrt{K^2 + m^2} - i\varepsilon = w_0 + m = w_3\]
\[w = -K + i\varepsilon = w_2\]
\[w = -\sqrt{K^2 + m^2} + i\varepsilon = w_0 + m = w_4\]

\[w_0 = \sqrt{K^2 + m^2}\]

The contribution from the large semicircle vanishes in the limit of infinite radius.
\[
\int_C = 2\pi i \left( \text{sum of residues of poles of integrand inside } C \right)
\]
\[
= 2\pi i \left\{ \frac{m-K}{(-2k)[\sqrt{k^2 m^2-m-K}]} \left[ -\sqrt{k^2 m^2-m-K} \right] \right\}_{\text{res. at } \omega_1} + 2\pi i \left\{ \frac{2m - \sqrt{k^2 m^2}}{(-2 \sqrt{k^2 m^2})[m-K - \sqrt{k^2 m^2}]} \left[ K^2 m^2 - \sqrt{k^2 m^2} \right] \right\}_{\text{res. at } \omega_2}
\]
\[
= 2\pi i \left\{ \frac{K-m}{4K^2 m} + \frac{1 - \frac{2m}{\sqrt{k^2 m^2}}}{-4m (\sqrt{k^2 m^2-m})} \right\}
\]
\[
2M \Delta m = \frac{e^2}{\pi^2} \times 4m, 2\pi i \int_0^\infty dk \left\{ \frac{K-m}{4m} \frac{1}{m} \right\}
\]
\[
= \frac{2e^2 i}{\pi} \int_0^\infty I(K) dK
\]
for small \( K \), \( I(K) \) is regular.

For large \( K \), \( I(K) \approx K-m - K \left( 1 - \frac{2m}{K} \right) \left( 1 + \frac{m^2}{K^2} + \frac{m^2}{2K^2} + \ldots \right) \)
\[
I(K) \approx K-m - K \left( 1 - \frac{2m}{K} \right) \left( 1 + \frac{m^2}{K^2} + \frac{m^2}{2K^2} + \ldots \right)
\]
\[
= K-m - K \left( 1 - \frac{3m^2}{2K^2} \right) = \frac{3m^2}{2K}
\]

ii. If we integrate from 0 to \( \Lambda \), then the leading term in \( \Lambda \) is
\[
\frac{2e^2 i}{\pi} \frac{3M^2}{2} \log \Lambda \quad \text{which is, as we asserted, logarithmically infinite.}
\]

Of course, other cutoff procedures are possible, and we consider now one in more detail.
The cutoff procedure we will adopt is relativistically invariant. It consists of modifying the propagator at high energies.

1. E. replace \( \frac{1}{k^2} \) by \( \frac{C(k^2)}{k^2} \).

Where \( C(k^2) \to 1 \) when \( k^2 \ll \Lambda^2 \)

we choose \( C(k^2) = -\frac{\Lambda^2}{k^2 - \Lambda^2} \to \frac{1}{k^2} \) as \( k^2 \to \infty \).

Suppose now that with this cutoff all integrals are mass convergent (they are not). If the answer for a problem depends on \( \Lambda \), we consider that we don't know the answer. But if as \( \Lambda \to \infty \), a finite limit ensures, then we consider the answer known in quantum electrodynamics.

**Discussion of the infinite correction to the electron mass**

\( m_p \), the mass that appears in the Dirac Equation is not the physical mass of the electron, because there are corrections to \( m_p \) due to the interaction of the electron with the electromagnetic field.

Theoretically, \( m_{\text{exp}}^2 = m_p^2 + \frac{3e^2}{\Lambda} m_p^2 (\log \frac{\Lambda^2}{m_e^2}) + O(e^4) \)

The problem is of course that \( \Delta m^2 = \infty \).

\( m_{\text{exp}} \) is the experimental, measured mass.

 Aside: If you look at the mass difference between the neutral and charged pion in the same way, you will obtain a quadratically divergent infinity.

The electromagnetic structure of the pion is simply not known.
Let's look at a problem that we can solve in higher order.

Calculate the shift in an energy level of the hydrogen atom due to the possibility of virtual emission and reabsorption of a photon scattered by the electron.

obtain \[ \Delta E_n = E_1 + \frac{i \Gamma}{2} \]

\( E_1, \Gamma \), are \textit{real} \quad \text{---} \quad \Gamma \) gives the lifetime of the state \quad \text{(rate of disintegration)}

ground state:
\[ \text{Re} \Delta E_0 = E_1 = \Delta m_p + \frac{mc^2}{8\pi^2} \]
the infinite correction to \( m_p \) to obtain \( m_{\text{exp}} \)

If you re-express the energy \textit{in terms of the physical mass (m_{\text{exp}})} rather than \( m_p \), then the correction is finite as \( \lambda \to \infty \).

There are ways of by-passing the infinities such as dispersion theory, but at the present time there is no method that is uniformly better than any other method for dealing with higher order corrections in qed.
Theory of $\beta$ Decay:

Ref. PR 109, 193 (1958), Feynman & Gell-Mann

There are 2 different kinds of neutrinos: $\nu_e, \nu_\mu$.

Muon decay occurs as: $\mu \rightarrow e + \bar{\nu}_e + \nu_\mu$.

The neutrinos are massless and obey Dirac's equation $\not{p} \psi_\nu = 0$ $p_\nu^2 = 0$

Muon decay, $\beta$ decay, and other process are described by the weak coupling:

$$\text{Rate} = \ldots \ 1 m^2$$

where $M$ is derived from a 4-particle point interaction

$$\chi \not{F} \chi$$

$$\chi a + c \rightarrow b + d$$

$$M = G \sqrt{8} \ (\bar{u}_d \bar{\nu}_e a \bar{u}_e) (\bar{u}_d \bar{\nu}_e a \bar{u}_e) = G \sqrt{8} (\bar{d} \bar{c})(\bar{b} \bar{a})$$

Abbreviated notation

$\sqrt{8}$ is there for historical reasons.

$$a = \frac{1 + i \gamma_5}{2}$$

$G$ is a constant with dimensions: $6M_P^2 = 1.01 \pm 0.01 \times 10^{-5}$ $M_P = \text{mass of proton}$

For particular weak processes, one must pick $a,b,c,d$ suitably.

For $\mu$ decay: $(\bar{e} \nu_e) (\bar{\nu}_\mu \mu)$

And that is the same as for $\nu_e + \mu \rightarrow \nu_\mu + e$
Problem: Calculate the shape of the energy spectrum in $\mu$ decay.

Aside: There is the IVB (intermediate vector boson) theory which says that $\mu$ decay: $\mu + \nu_e \rightarrow e + \nu_\mu$, for example, is

\[
\begin{array}{c}
e \\ \mu \\ \nu_e
\end{array}
\quad \text{rather than}
\begin{array}{c}
e \\ \nu_e
\end{array}
\]

I.e., that the interaction is mediated by a vector boson, but that the point interaction description is a good approximation because the IVB (called the $W$ particle) is massive, and therefore corresponds to a short range force.

The interaction is written $\mathcal{L} = \frac{G_\mu}{\sqrt{2}} J_\mu J_\mu$

where $J_\mu = (\bar{\nu}_e \gamma_\mu \gamma_5 \nu_\mu) + (\bar{\mu} \gamma_\mu \gamma_5 \mu_e) + J_\mu^{\text{hadrons}}$

$J_\mu^{\text{hadron}} = \text{weak current of strongly interacting particles} (u, p, n, \pi, K, \ldots)$

Problem: Calculate the rate of any weak process that interests you.

Aside: One can also do electromagnetic corrections to weak processes. For example, in $\mu$ decay, neutron $\beta$ decay

\[
\begin{array}{c}
\nu_e \\ \mu \\ \nu_n
\end{array}
\]

there is the diagram

\[
\begin{array}{c}
\nu_e \\ \mu_n \
\end{array}
\]

\[
\begin{array}{c}
\nu_e \\ \mu_n \
\end{array}
\]
Returning now to the problem of infinities in QED, we write

\[ C(\ell^2) = \frac{-\Lambda^2}{\ell^2 - (\ell^2 - \Lambda^2) \ell^2} = \frac{1}{\ell^2} - \frac{1}{\ell^2 - \Lambda^2} \]

the first term is the one obtained from a massless photon,
the second is obtained from a photon of mass \( \Lambda \) that couples
to an imaginary charge. The charge is imaginary because
what was \( +ie^2 \) is now \( -ie^2 \). The imaginary charge
implies that the Hamiltonian is non-Hermitian, and so
unitarity is violated.

In the limit \( \Lambda \to \infty \), things seem to be OK, but that has
never been proved.

I do not believe that everything has been straightened out.
The readers always have a weak point in their arguments
where they cannot prove anything.

Question: Is the final answer consistent with unitarity?

The procedures \( \{ \text{Mad.} \to \text{Mexp.} \} \) in all calculations
do not straighten
\[ \frac{1}{\ell^2} \to \frac{C(\ell^2)}{\ell^2} \]
not straighten out all infinities.

Ex. e-e scattering in 4th order

\[ M = \]

The 4th order diagrams give a contribution
\[ \epsilon^2 (1 + e^2 \log \Lambda') \delta \mu \ldots \delta \mu \]
If then we identify
\[ e^2_{\text{exp}} = e^2_{\text{th}} \left( 1 + e^2_{\text{th}} \frac{\log N'}{m^2_{\text{th}}} + \ldots \right) \]
and say that \( e^2_{\text{exp}} \) observed charge
\[ e^2_{\text{th}} = \text{unrenormalized ("theoretical") charge} \]
we once again have an infinite renormalization.

If now we find the rate, in any order, of a process
\[ \Gamma = \Gamma (m_0, e_{\text{th}}, N, N') = G (m_{\text{exp}}, e_{\text{exp}}, N, N') \]
then \( \lim \limits_{N \to \infty} G (m_{\text{exp}}, e_{\text{exp}}, N, N') \) exists.

The process of finding \( m_{\text{exp}} = m_{\text{exp}} (m_0, \Lambda) \)
and \( e_{\text{exp}} = e_{\text{exp}} (e_{\text{th}}, \Lambda') \)

is known as mass and charge renormalization.