

1. (a) According to the page 236 of the note, the shifted frequency for strings with two ends fixed can be written as

$$\Omega_n = \omega_n \left(1 - \frac{1}{\rho_0 l} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx \right)$$

In this case, the perturbed density is

$$\rho_1(x) = M \delta(x-b)$$

Thus, above formula gives

$$\Omega_n = \frac{n\pi c}{l} \left(1 - \frac{M}{\rho_0 l} \sin^2 \frac{n\pi b}{l} \right) = \frac{n\pi c}{l} \left(1 - \frac{M}{m} \sin^2 \frac{n\pi b}{l} \right)$$

where we used

$$\omega_n = \frac{n\pi c}{l}, \quad \rho_0 l = m$$

2. (a) We should solve the wave equations $(c_{1,2} = \sqrt{T(\rho_0 \pm \epsilon)})$

$$\frac{1}{c_1^2} \frac{\partial^2}{\partial x^2} S = \frac{\partial^2}{\partial x^2} S$$

for $0 < x < b$, and

$$\frac{1}{c_2^2} \frac{\partial^2}{\partial x^2} S = \frac{\partial^2}{\partial x^2} S$$

for $b < x < l$ with the boundary conditions,

$$S(0) = 0, \quad S(l) = 0$$

Following the conventional methods, the solution for the above equations can be written as,

$$S(x) = \begin{cases} A \sin \frac{\Omega}{c_1} x \cos \Omega(t-t_0) & 0 < x < b \\ B \sin \frac{\Omega}{c_2} (l-x) \cos \Omega(t-t_0) & b < x < l \end{cases}$$

Since the string should be connected at $x=b$, we should have the continuity of s .

Furthermore, since there is no mass attached at the connection point, the transverse component of the tension should be continuous. This implies the derivative of s at $x=b$ is continuous. Consequently, we have

$$\begin{aligned} \frac{S'}{S} \Big|_{b+\epsilon} &= \frac{S'}{S} \Big|_{b-\epsilon} \Rightarrow \frac{\frac{\Omega}{c_1} \cos \frac{\Omega}{c_1} x / \sin \frac{\Omega}{c_1} x}{\sin \frac{\Omega}{c_1} x} \Big|_{x=b-\epsilon} = -\frac{\Omega}{c_2} \cos \frac{\Omega}{c_2} (l-x) / \sin \frac{\Omega}{c_2} (l-x) \Big|_{x=b+\epsilon} \\ &\Rightarrow c_1 \tan \frac{\Omega}{c_1} b = -c_2 \tan \frac{\Omega}{c_2} (l-b) \end{aligned}$$

- (b) Using the same formula as used in problem 1, we have, $(\omega_n = \frac{n\pi c}{l})$

$$\Omega_n \sim \frac{n\pi c}{l} \left(1 - \frac{1}{\rho_0 l} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx \right)$$

The perturbed density in this case can be written as,

$$\rho_1(x) = \begin{cases} \epsilon & 0 < x < b \\ -\epsilon & b < x < l \end{cases}$$

Thus, the direct integration gives,

$$\begin{aligned} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx &= \epsilon \int_0^b \sin^2 \frac{n\pi x}{l} dx - \epsilon \int_b^l \sin^2 \frac{n\pi x}{l} dx \\ &= \frac{\epsilon}{2} \left\{ \int_0^b \left(1 - \cos \frac{2n\pi x}{l} \right) dx - \int_b^l \left(1 - \cos \frac{2n\pi x}{l} \right) dx \right\} = \epsilon \left\{ b - \frac{l}{2} - \frac{l}{2n\pi} \sin \frac{2n\pi b}{l} \right\} \end{aligned}$$

This shows that

$$\Delta \Omega_n \sim -\omega_n \cdot \frac{\epsilon}{\rho_0 l} \left(b - \frac{l}{2} - \frac{l}{2n\pi} \sin \frac{2n\pi b}{l} \right)$$

We can immediately see that the above expression vanishes as we set $b = l/2$.

$$(\because \sin n\pi = 0)$$

3. For transverse oscillation, the kinetic energy can be written as,

$$T = \frac{1}{2} \int dm \cdot \dot{s}^2$$

whereas the potential energy can be written as, (dl ; length element)

$$U = T \int dl$$

where s denotes the transverse displacement and we do not care about the constant term appearing in the potential energy. In spherical polar coordinate, the length element can be written as (cf. Notice the definition of θ)

$$dl^2 = a^2 d\theta^2 + a^2 \cos^2 \theta d\phi^2$$

and the mass element dm is

$$dm = a \cdot d\phi \cdot \rho$$

and the transverse displacement is.

$$s = a \sin \theta$$

where we refer to the right figure to

get the result. Thus, the total Lagrangian can be rewritten as, ($\dot{\theta} = \frac{d}{dt} \theta$, $\theta' = \frac{d}{d\phi} \theta$)

$$\begin{aligned} L &= \int \frac{1}{2} \cdot \rho a d\phi (a \cos \theta \dot{\theta})^2 - T a \int \sqrt{d\theta^2 + \cos^2 \theta d\phi^2} \\ &= \int_0^{2\pi} \left(\frac{1}{2} \rho a^3 \cos^2 \theta \dot{\theta}^2 - T a \sqrt{\theta'^2 + \cos^2 \theta} \right) d\phi \end{aligned}$$

Since we are considering small oscillation, we expand the above expression to retain terms only up to the second order in θ . Then, $(\cos \theta = 1 - \frac{\theta^2}{2} + \dots, \cos^2 \theta = 1 - \theta^2 + \dots)$

$$L = \int_0^{2\pi} \mathcal{L} d\phi = \int_0^{2\pi} \left(\frac{1}{2} \rho a^3 \dot{\theta}^2 - T a \left(1 - \frac{\theta^2}{2} + \frac{\theta'^2}{2} \right) \right) d\phi$$

Thus, the Euler-Lagrange equation gives,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) + \frac{d}{d\phi} \left(\frac{\partial \mathcal{L}}{\partial \theta'} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \rho a^3 \ddot{\theta} - T a \theta'' - T a \theta = 0$$

By assuming the solution of the form

$$\theta = \cos n\phi \cos \omega t$$

we have

$$-\rho a^3 \omega^2 + T a n^2 - T a = 0 \Rightarrow \omega^2 = \frac{T}{\rho a^2} (n^2 - 1)$$

4. (a) For small transverse oscillation, the string is in equilibrium approximately, as long as the vertical force balance is concerned. Thus, if x is measured upwards from the bottom end of the string,

tension at x , $T(x)$ = gravitational weight of the string hanged below $x = \rho g x$

where ρ is the line density. Since the transverse component of tension at point x

is $T \frac{ds}{dx} = T s'$, the equation of motion is given by

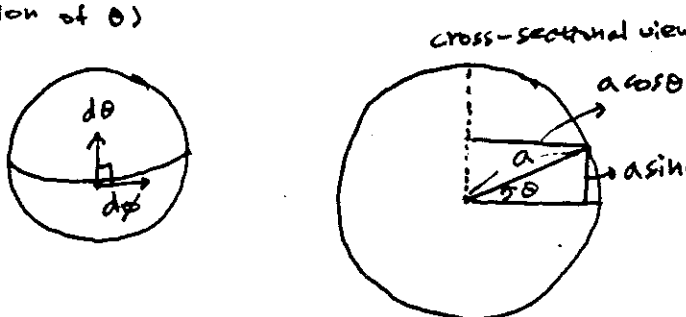
$$\rho \ddot{s} dx = -T(x) s'(x) + T(x+dx) s'(x+dx) = (T' s' + T s'') dx$$

$$\therefore x g s'' + g s' - \ddot{s} = 0 \quad (1)$$

where s denotes the transverse displacement. By setting

$$s = f(x) \cos \omega t$$

(1) reduces to



$$x \frac{d^2}{dx^2} f + \frac{d}{dx} f + \frac{\omega^2}{g} f = 0 \quad (2)$$

Since $x = z^2$, we have $dx = 2z dz$ which implies,

$$\frac{d}{dx} = \frac{1}{2z} \frac{d}{dz}$$

Thus, (2) becomes,

$$z^2 \cdot \frac{1}{2z} \frac{d}{dz} \left(\frac{1}{2z} \frac{d}{dz} f \right) + \frac{1}{2z} \frac{d}{dz} f + \frac{\omega^2}{g} f = \frac{1}{4} \frac{d^2}{dz^2} f + \frac{1}{4z} \frac{d}{dz} f + \frac{\omega^2}{g} f = 0$$

$$\therefore \frac{d^2}{dz^2} f + \frac{1}{z} \frac{d}{dz} f + \frac{4\omega^2}{g} f = 0$$

(b) In this case, relevant expressions for the kinetic energy and the potential energy which are valid in small transverse oscillation is,

$$K.E = \frac{1}{2} \int \rho dx \dot{s}^2$$

$$P.E = \frac{1}{2} \int T \left(\frac{ds}{dx} \right)^2 dx \quad (\text{cf. } \sqrt{1+s'^2} dx - dx \approx \frac{1}{2} s'^2 dx)$$

Thus, their time average values are

$$\langle K.E \rangle = \frac{1}{4} \rho \omega^2 \int_0^l f(x)^2 dx$$

$$\langle P.E \rangle = \frac{1}{4} \int_0^l T \left(\frac{df}{dx} \right)^2 dx$$

for $s = f(x) \cos \omega t$. As an approximate trial function which satisfies boundary condition $f(l) = 0$, we choose

$$f(x) = l^p - x^p$$

Then, the above equations become,

$$\langle K.E \rangle = \frac{1}{4} \rho \omega^2 \int_0^l (l^p - x^p)^2 dx = \frac{p^2}{2(p+1)(2p+1)} \rho \omega^2 l^{2p+1}$$

$$\langle P.E \rangle = \frac{1}{4} \cdot g \rho \cdot \int_0^l x (p x^{p-1})^2 dx = \frac{p g}{8} p \cdot l^{2p}$$

By equating the two expressions, we have

$$\omega^2 = \frac{(p+1)(2p+1)}{4p} \frac{g}{l} \equiv h(p) \frac{g}{l}$$

Now, $\frac{d}{dp} h(p) = 0$ gives,

$$(4p+3) \cdot 4p - 4(p+1)(2p+1) = 4(2p^2 - 1) = 0 \quad \therefore p = \frac{1}{\sqrt{2}}$$

where we should have positive p to have regular f at $x = 0$. Thus,

$$\omega = \sqrt{h\left(\frac{1}{\sqrt{2}}\right)} \cdot \sqrt{\frac{g}{l}} \approx 1.207 \sqrt{\frac{g}{l}}$$

5. As explained in the problem 4, the tension should balance the centrifugal force.

Thus,

$$T(x) = \int_x^l \rho r \Omega^2 dx = \frac{\rho}{2} (l^2 - x^2) \Omega^2$$

Thus, the equation of motion becomes,

$$\rho \ddot{s} dx = -T(x) s'(x) + T(x+dx) s'(x+dx) \approx (T' s' + T s'') dx$$

$$\rho \ddot{s} = -\rho \Omega^2 x s' + \frac{\rho}{2} \Omega^2 (l^2 - x^2) s''$$

If we set $s = f(x) \cos \omega t$, then the above equation becomes,

$$-\frac{2\omega^2}{\Omega^2} f = -2x \frac{df}{dx} + (l^2 - x^2) \frac{d^2 f}{dx^2} = \frac{d}{dx} (l^2 - x^2) \frac{df}{dx}$$

By introducing $z = x/l$, above equation can be rewritten as,

$$\frac{d}{dz} (1-z^2) \frac{df}{dz} + \frac{2\omega^2}{\Omega^2} f = 0$$

6. From the note, the equation of the motion in this case is, (neglecting rotational K.E.)

$$\ddot{s} + (cd)^2 s'''' = 0$$

where $c^2 = AY/p$ and $d^2 = I/pA^2$. In static case, above equation reduces to

$$s'''' = 0$$

For $x > b$, the bar is assumed to be straight. For $x < b$, considering the clamped boundary condition at $x = 0$, i.e., $s(0) = s'(0) = 0$, the solution is of the form,

$$s = \alpha x^3 + \beta x^2 \quad x \leq b$$

For the smooth shape of the bar, (notice that the governing equation is the fourth order differential equation), we require that s , s' and s'' be continuous at $x = b$. If we write $s = px + q$ as $x > b$ case solution, the resulting conditions are

$$\alpha b^3 + \beta b^2 = pb + q$$

$$3\alpha b^2 + 2\beta b = p$$

$$6\alpha b + 2\beta = 0$$

which can be easily solved to yield,

$$\beta = -3b\alpha, \quad p = -3b^2\alpha, \quad q = b^3\alpha$$

Without the loss of generality we can set $\alpha = -1$. Then,

$$s(x) = \begin{cases} 3bx^2 - x^3 & 0 < x < b \\ 3b^2x - b^3 & b < x < l \end{cases} \quad (1)$$

To use Rayleigh's method, we assume the solution of the form

$$s(x,t) = f(x) \cos \omega t$$

where $f(x)$ is assumed to be (1), the position of bar in static case. Now the averaged kinetic energy is given by

$$\begin{aligned} \langle K.E \rangle &= \left\langle \frac{1}{2} pA \int dx \dot{s}^2 \right\rangle = \frac{1}{4} pA \omega^2 \int_0^l dx f(x)^2 \\ &= \frac{1}{4} pA \omega^2 \left\{ \int_0^b (3bx^2 - x^3)^2 dx + \int_b^l (3b^2x - b^3)^2 dx \right\} \\ &= \frac{1}{4} pA \omega^2 \left(-\frac{2}{3} b^7 + b^6 l - 2b^4 l^2 + 3b^4 l^3 \right) \end{aligned} \quad (2)$$

Neglecting the rotational kinetic energy, the remaining potential energy becomes,

$$\begin{aligned} \langle P.E \rangle &= \left\langle \frac{YI}{2p} \int dx \left(\frac{ds}{dx} \right)^2 \right\rangle = \frac{YI}{4p} \int_0^l dx \left(\frac{df}{dx} \right)^2 \\ &= \frac{YI}{4p} \int_0^b (6b - 6x)^2 dx \\ &= \frac{3YI}{p} b^3 \end{aligned} \quad (3)$$

By equating (2) and (3), we have,

$$\frac{1}{\omega^2} = \frac{p^2 A}{12 Y I} \left(-\frac{2}{3} b^4 + b^3 l - 2b^2 l^2 + 3b l^3 \right) = \frac{2l^4}{c^2 d^2} \cdot \frac{1}{12} \cdot \left(-\frac{2}{3} r^4 + r^3 - 3r^2 + 3r \right)$$

where $r = b/l$. The value of b can be determined by,

$$\frac{d}{dr} \left(\frac{1}{\omega^2} \right) = 0 \Rightarrow -\frac{8}{3} b^3 + 3b^2 l - 6b l^2 + 3l^3 = 0$$

As an approximation, we assume $r = 3/4$. Then above expression can be written as,

$$\omega = \frac{c \cdot d}{l^2} \sqrt{\frac{12}{-\frac{2}{3} \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^3 - 3\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right)}} \approx 3.524 \frac{cd}{l^2}$$

7. In general case, the length element can be written as,

$$ds^2 = (r_0^2 d\theta)^2 = (r_0 + \delta r)^2 d(\theta + \delta\theta)^2 + (d\delta r)^2$$

where we used the fact that the length element should be $ds = r_0 d\theta$. Neglecting second order terms, we get,

$$\delta r d\theta + r_0 d\delta\theta = 0 \Rightarrow \frac{\delta r}{r_0} = -\frac{d}{d\theta} \delta\theta \quad (1)$$

From p.239 of note, the potential energy can be written as,

$$V = \frac{\gamma I}{2\rho} \int \frac{1}{R^2} ds$$

where ds is the length element given by $ds = r_0 d\theta$. Using the expression for curvature,

$$\frac{1}{R} = \frac{1}{r_0 + \delta r} + \frac{d^2}{d\theta^2} \left(\frac{1}{r_0 + \delta r} \right) = \frac{1}{r_0} \left(1 - \frac{\delta r}{r_0} + \frac{d^2}{d\theta^2} \frac{\delta r}{r_0} \right) = \frac{1}{r_0} (1 + \delta\theta' + \delta\theta''')$$

where we retained only up to linear order term and used (1), the aforementioned potential energy becomes,

$$V = \frac{\gamma I}{2\rho r_0} \int (1 + \delta\theta' + \delta\theta''')^2 d\theta$$

If we neglect the rotational kinetic energy, the kinetic energy can be written as,

$$\begin{aligned} T &= \frac{1}{2} \rho A \int ds (\dot{\delta r}^2 + r_0^2 \dot{\delta\theta}^2) \\ &= \frac{1}{2} \rho A r_0^3 \int d\theta \left(\left(\frac{\delta r}{r_0}\right)^2 + \dot{\delta\theta}^2 \right) \\ &= \frac{1}{2} \rho A r_0^3 \int (\delta\theta')^2 + (\delta\ddot{\theta})^2 d\theta \end{aligned}$$

where we used (1). Thus, the total Lagrangian can be written as,

$$L = \int \left[\frac{1}{2} \rho A r_0^3 (\delta\ddot{\theta}^2 + (\delta\theta')^2) - \frac{\gamma I}{2\rho r_0} (1 + \delta\theta' + \delta\theta''')^2 \right] d\theta \quad (2)$$

The variation of the action can be written as, $(L = \int \mathcal{L} d\theta)$

$$\begin{aligned} \delta \mathcal{I} &= \int \delta L d\theta = \int \delta \mathcal{L} d\theta = \int \left(\frac{\partial \mathcal{L}}{\partial \delta\theta} \delta\theta + \frac{\partial \mathcal{L}}{\partial \delta\dot{\theta}} \delta\dot{\theta} + \frac{\partial \mathcal{L}}{\partial \delta\theta'} \delta\theta' + \frac{\partial \mathcal{L}}{\partial \delta\theta''} \delta\theta'' + \frac{\partial \mathcal{L}}{\partial \delta\theta'''} \delta\theta'' \right) \\ &= \int \left\{ \frac{\partial \mathcal{L}}{\partial \delta\theta} - \frac{\partial}{\partial \theta} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta'} \right) + \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta''} \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta'''} \right) - \frac{\partial^3}{\partial \theta^3} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta'''} \right) \right\} \delta\theta d\theta \end{aligned}$$

where we used integration by parts and assumed proper boundary conditions. Above equation tells us that the equation of motion can be written as,

$$\frac{\partial}{\partial \theta} \left(\frac{\partial \mathcal{L}}{\partial \delta\dot{\theta}} \right) - \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta'} \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta''} \right) + \frac{\partial^3}{\partial \theta^3} \left(\frac{\partial \mathcal{L}}{\partial \delta\theta'''} \right) = 0$$

From (2), this equation yields,

$$\frac{\partial}{\partial \theta} (\delta\ddot{\theta}) - \frac{\partial^2}{\partial \theta^2} (\delta\theta') - \frac{\gamma I}{\rho^2 A r_0^4} \left\{ \frac{\partial}{\partial \theta} (1 + \delta\theta' + \delta\theta''') + \frac{\partial^3}{\partial \theta^3} (1 + \delta\theta' + \delta\theta''') \right\} = 0$$

$$\delta\ddot{\theta} - \delta\theta'' - \frac{\gamma I}{\rho^2 A r_0^4} (\delta\theta'' + 2\delta\theta'''' + \delta\theta''''') = 0$$

By putting

$$\delta\theta = e \cos n\theta \cos \omega t$$

into (3), we have

$$-\omega^2 - \omega^2 n^2 - \frac{\gamma I}{\rho^2 A r_0^4} (-n^2 + 2n^4 - n^6) = 0$$

which can be rearranged to yield,

$$\omega^2 = \frac{\gamma I}{\rho^2 A r_0^4} \frac{n^2(n^2-1)^2}{n^2+1}$$

8. In case of the square drum, the kinetic and the potential energy are given by

$$K.E = \frac{1}{2} \int \rho \dot{s}^2 dx dy$$

$$P.E = \frac{T}{2} \int \left\{ \left(\frac{\partial s}{\partial x} \right)^2 + \left(\frac{\partial s}{\partial y} \right)^2 \right\} dx dy$$

By setting $s = f(x,y) \cos \omega t$, where $f(x,y)$ denotes the spatial normal mode solution, the time-averaging gives

$$\langle K.E \rangle = \frac{1}{4} \rho \omega^2 \int f^2(x,y) dx dy$$

$$\langle P.E \rangle = \frac{T}{4} \int \left\{ \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right\} dx dy$$

For the lowest three frequencies, the proper normal modes are

$$f_{1,1} = \sin\left(\frac{\pi}{l}x\right) \sin\left(\frac{\pi}{l}y\right)$$

$$f_{2a} = \left(\cos\frac{\pi a}{l} \sin\frac{2\pi x}{l} \sin\frac{\pi y}{l} - \cos\frac{\pi b}{l} \sin\frac{\pi x}{l} \sin\frac{2\pi y}{l} \right) / \sqrt{\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l}}$$

$$f_{2b} = \left(\cos\frac{\pi a}{l} \sin\frac{\pi x}{l} \sin\frac{2\pi y}{l} + \cos\frac{\pi b}{l} \sin\frac{2\pi x}{l} \sin\frac{\pi y}{l} \right) / \sqrt{\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l}}$$

Notice that in f_{2a} , M lies on a node of the oscillation and f_{2b} is orthogonal to the mode. Using the integrals,

$$\int_0^l \cos^2 \frac{n\pi x}{l} dx = \int_0^l \sin^2 \frac{n\pi x}{l} dx = \frac{l}{2}, \quad \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0 \text{ if } n \neq m$$

the potential energy part can be immediately calculated. The results are

$$\langle P.E \rangle_{1,1} = \frac{T}{4} \cdot \frac{l}{2} \cdot \frac{l}{2} \cdot 2 \left(\frac{\pi}{l} \right)^2$$

$$\langle P.E \rangle_{2a} = \langle P.E \rangle_{2b} = \frac{T}{4} \cdot \frac{l}{2} \cdot \frac{l}{2} \cdot 5 \left(\frac{\pi}{l} \right)^2$$

(This result is clear recalling the orthogonality) The density can be put into the form,

$$\rho = \rho_0 + M \delta(x-a) \delta(y-b)$$

Since the first of the above two terms is constant, the contribution to kinetic energy by this term ($\langle K.E \rangle^{\delta}$) can also be trivially calculated to yield,

$$\langle K.E \rangle_{\delta,1}^{\delta} = \frac{1}{4} \rho_0 \omega^2 \cdot \frac{l}{2} \cdot \frac{l}{2}$$

$$\langle K.E \rangle_{\delta,2a}^{\delta} = \langle K.E \rangle_{\delta,2b}^{\delta} = \frac{1}{4} \rho_0 \omega^2 \cdot \frac{l}{2} \cdot \frac{l}{2}$$

Now, the contribution from the second term ($\langle K.E \rangle^M$) is,

$$\langle K.E \rangle_{1,1}^M = \frac{\omega^2}{4} \int M \delta(x-a) \delta(y-b) \left(\sin\frac{\pi}{l}x \sin\frac{\pi}{l}y \right)^2 dx dy = \frac{\omega^2}{4} M \cdot \sin^2\frac{\pi}{l}a \sin^2\frac{\pi}{l}b$$

$$\langle K.E \rangle_{2a}^M = 0 \quad (\because M \text{ lies on the node})$$

$$\langle K.E \rangle_{2b}^M = \frac{\omega^2}{4} \int M \delta(x-a) \delta(y-b) f_{2b}^2(x,y) = \frac{\omega^2}{4} M f_{2b}^2(a,b) = \frac{\omega^2}{4} M \cdot 4 \sin^2\frac{\pi}{l}a \sin^2\frac{\pi}{l}b \left(\cos^2\frac{\pi}{l}a + \cos^2\frac{\pi}{l}b \right)$$

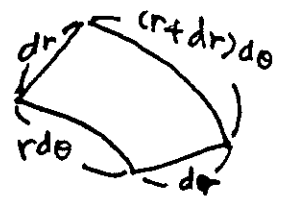
Thus, the three frequencies are, (setting $\langle K.E \rangle = \langle P.E \rangle$)

$$\left\{ \begin{aligned} \Omega_{1,1}^2 &= 1 + \frac{4M}{\rho_0 l^2} \sin^2\frac{\pi a}{l} \sin^2\frac{\pi b}{l} = \frac{T}{\rho} 2 \left(\frac{\pi}{l} \right)^2 = \omega_{1,1}^2 \\ \Omega_{2a}^2 &= \frac{T}{\rho} \cdot 5 \left(\frac{\pi}{l} \right)^2 = \omega_{1,2}^2 = \omega_{2,1}^2 \\ \Omega_{2b}^2 &= 1 + \frac{16M}{\rho_0 l^2} \sin^2\frac{\pi a}{l} \sin^2\frac{\pi b}{l} \left(\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l} \right) = \frac{T}{\rho} \cdot 5 \left(\frac{\pi}{l} \right)^2 = \omega_{2,1}^2 \end{aligned} \right.$$

$$\Rightarrow \left\{ \begin{aligned} \Omega_{1,1} &= \omega_{1,1} \left(1 - \frac{2M}{\rho_0 l^2} \sin^2\frac{\pi a}{l} \sin^2\frac{\pi b}{l} \right) \\ \Omega_{2a} &= \omega_{1,2} = \omega_{2,1} \\ \Omega_{2b} &= \omega_{2,1} \left(1 - \frac{8M}{\rho_0 l^2} \sin^2\frac{\pi a}{l} \sin^2\frac{\pi b}{l} \left(\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l} \right) \right) \end{aligned} \right.$$

9. Let ρ denote the surface mass density and T , surface tension.

Referring to the right figure, we can write equation of the motion as follows,



$\rho \ddot{s} r dr d\theta = T (r + dr) d\theta \frac{\partial}{\partial r} s(r + dr) - T r d\theta \frac{\partial}{\partial r} s(r) + T dr (\frac{\partial}{\partial \theta} s(\theta + d\theta) - \frac{\partial}{\partial \theta} s)$
 just as the case in p. 245 of the note. Neglecting third order or higher order terms, we get,

$$\rho \ddot{s} r dr d\theta = T dr d\theta (\frac{\partial}{\partial r} s + r \frac{\partial^2}{\partial r^2} s + \frac{\partial^2}{\partial \theta^2} s)$$

By defining $c^2 = T/\rho$, we have

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} s = \frac{\partial^2}{\partial r^2} s + r \frac{\partial}{\partial r} s + \frac{\partial^2}{\partial \theta^2} s$$

We set

$$s = f(r) g(\theta) h(t)$$

and require that

$$\frac{d^2}{dt^2} g(\theta) = -n^2 g, \quad \frac{d^2}{dt^2} h = -\omega^2 h \quad (1)$$

Then the resulting equation for $f(r)$ is,

$$\frac{d^2}{dr^2} f(r) + \frac{1}{r} \frac{d}{dr} f(r) + (\frac{\omega^2}{c^2} - \frac{n^2}{r^2}) f(r) = 0$$

Since this equation contains of r variable, we see that the equation is separated satisfactorily. Eqs. (1) can easily be solved to yield,

$$h = \cos \omega t \text{ or } \sin \omega t$$

$$g = \cos n\theta \text{ or } \sin n\theta$$

In g , since the function should be continuously matched at $\theta=0$ and $\theta=2\pi$, we should require that n be an integer.

The time-averaged kinetic energy and the potential energy is (for $n=0$ case)

$$\langle K.E \rangle = \langle \frac{1}{2} \int \rho \dot{s}^2 dx dy \rangle = \frac{\rho \omega^2}{4} \int f^2 r dr d\theta = \frac{\pi}{2} \rho \omega^2 \int_0^a f^2 r dr$$

$$\langle P.E \rangle = \langle \frac{T}{2} \int \{ (\frac{\partial}{\partial r} s)^2 + (\frac{\partial}{\partial \theta} s)^2 \} dx dy \rangle = \frac{\pi}{2} T \cdot \int_0^a (\frac{d}{dr} f)^2 r dr$$

As a trial function, we can consider

$$f(r) = a^p - r^p$$

which satisfies the boundary conditions, $f(a)=0$ and $f'(0)=0$ provided that $p > 0$. By

the direct integrations, we have,

$$\langle K.E \rangle = \frac{\pi}{4} \rho \omega^2 \frac{p^2}{(p+2)(p+1)} a^{2p+2}$$

$$\langle P.E \rangle = \frac{\pi}{4} T p \cdot a^{2p}$$

By setting the two quantities we have,

$$\omega^2 = \frac{(p+1)(p+2)}{p} \frac{T}{\rho} \cdot \frac{1}{a^2} = \frac{(p+1)(p+2)}{p} (\frac{c}{a})^2 \equiv g(p) (\frac{c}{a})^2$$

To minimize ω , we require that $\frac{d}{dp} g(p) = 0$, which yields,

$$(2p+3)p - (p^2 + 3p + 2) = p^2 - 2 = 0 \Rightarrow p = \pm \sqrt{2} \text{ (since } p > 1), \quad \underline{p = \sqrt{2}}$$

Thus,

$$\omega = \sqrt{g(\frac{\sqrt{2}}{\sqrt{2}})} \frac{c}{a} = \sqrt{\frac{(4\sqrt{2})(2+\sqrt{2})}{\sqrt{2}}} \frac{c}{a} \approx 2.414 \frac{c}{a}$$

10. The total energy of this system can be written as,

$$E = \int_0^l \left(\frac{YI}{2P} (s'')^2 - Mg \cdot s \cdot \delta(x-x_0) \right) dx = \int_0^l \mathcal{L} dx$$

Thus, the Euler-Lagrange equation gives,

$$\frac{d^2}{dx^2} \left(\frac{\partial \mathcal{L}}{\partial s''} \right) + \frac{\partial \mathcal{L}}{\partial s} = 0$$

$$\Rightarrow \frac{YI}{P} s'''' - Mg \delta(x-x_0) = 0 \quad (1)$$

Since the bar is supported at its ends, the solution can be Fourier-expanded as follows.

$$s = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x$$

Putting this expression into (1), we have

$$\sum_{n=1}^{\infty} \left(\frac{YI}{P} A_n \left(\frac{n\pi}{l} \right)^4 \sin \frac{n\pi x}{l} \right) - Mg \delta(x-x_0) = 0$$

We take $\int_0^l \sin \frac{n\pi x}{l} dx$ to the above equation. Then,

$$\frac{l}{2} \cdot \frac{YI}{P} A_n \left(\frac{n\pi}{l} \right)^4 = Mg \sin \frac{n\pi}{l} x_0 \Rightarrow A_n = \frac{2Mg}{\pi^4 Y} \left(\frac{l^3}{I} \right) \sin \frac{n\pi}{l} x_0 \cdot \frac{1}{n^4}$$

Referring to the right figure, I/P can be calculated as follows.

$$\frac{I}{P} = \int_{-h/2}^{h/2} y^2 dy \omega = \frac{h^3 \omega}{12}$$

Thus, the expansion coefficients are

$$A_n = \frac{24Mg}{\pi^2 Y} \left(\frac{l^3}{\omega h^3} \right) \sin \frac{n\pi}{l} x_0 \cdot \frac{1}{n^4}$$

Thus, the full solution is,

$$s = \frac{24Mg}{\pi^2 Y} \left(\frac{l^3}{\omega h^3} \right) \sum_{n=1}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l}$$

