PH205, Problem Set 11.

1. Spinning Basketballs

The constraint condition of rolling wihtout slipping can be written as

$$\vec{v} + \vec{\omega}\times\vec{a} = 0 \qquad (1)$$

From the geometry of this problem, the velocity vector and the angular velocity vector can be written as

$$\vec{v} = (a+b)\frac{d}{dt}\hat{1}, \quad \vec{\omega} = \omega_1\hat{1} + \vec{\omega}_\perp \quad (\vec{\omega}_\perp\cdot\hat{1} = 0)$$

Thus, $\hat{1}\times(1)$ gives ($\vec{a} = -a\hat{1}$)

$$(a+b)\,\hat{1}\times\frac{d}{dt}\hat{1} + \hat{1}\times(\vec{\omega}\times\vec{a}) = (a+b)\,\hat{1}\times\frac{d}{dt}\hat{1} - a\vec{\omega}_\perp = 0$$

$$\therefore \vec{\omega} = \omega_1\hat{1} + \frac{a+b}{a}\,\hat{1}\times\frac{d}{dt}\hat{1} \qquad (2)$$

The force and torque equations are given by,

$$\vec{F} - mg\hat{z} = m\frac{d}{dt}\vec{v}$$

$$\vec{a}\times\vec{F} = \frac{d}{dt}(\overleftrightarrow{I}\cdot\vec{\omega}) = I\frac{d}{dt}\vec{\omega}$$

where we used the spherical symmetry to reduce $\overleftrightarrow{I}\cdot\vec{\omega} = I\vec{\omega}$. Thus, combining these two equations yields,

$$I\frac{d}{dt}\vec{\omega} = \vec{a}\times\left(mg\hat{z} + m\frac{d}{dt}\vec{v}\right) = \vec{a}\times\left(mg\hat{z} + m\frac{d}{dt}(\vec{a}\times\vec{\omega})\right)$$

$$I\frac{d}{dt}\left(\omega_1\hat{1} + \frac{a+b}{a}\,\hat{1}\times\frac{d}{dt}\hat{1}\right) = mga(-\hat{1}\times\hat{z}) + ma^2\,\hat{1}\times\frac{d}{dt}\left(\hat{1}\times\left(\omega_1\hat{1} + \frac{a+b}{a}\,\hat{1}\times\frac{d}{dt}\hat{1}\right)\right)$$

where we used (2) as $\vec{\omega}$. Using $\frac{d}{dt}\hat{1}\times\frac{d}{dt}\hat{1} = 0$, $\hat{1}\times\hat{1} = 0$, and $\hat{1}\times(\hat{1}\times\frac{d}{dt}\hat{1}) = -\frac{d}{dt}\hat{1}$ which is valid since $\hat{1}$ and $\frac{d}{dt}\hat{1}$ are perpendicular to each other, we have

$$(I+ma^2)\frac{a+b}{a}\,\hat{1}\times\frac{d^2}{dt^2}\hat{1} + I\omega_1\frac{d}{dt}\hat{1} + mga(\hat{1}\times\hat{z}) = 0 \qquad (3)$$

since

$$\vec{\omega} = \dot{\phi}\hat{z} - \dot{\theta}\frac{\hat{1}\times\hat{z}}{\sin\theta}$$

we have

$$\frac{d}{dt}\hat{1} = \vec{\omega}\times\hat{1} = -\dot{\phi}\,\hat{1}\times\hat{z} + \dot{\theta}\frac{\sin\theta}{\cos\theta}\hat{1} - \frac{\dot{\theta}}{\sin\theta}\hat{z}$$

The similar calculations give

$$\frac{d^2}{dt^2}\hat{1} = \left(\ddot{\theta}\frac{\cos\theta}{\sin\theta} - \dot{\theta}^2 - \dot{\phi}^2\right)\hat{1} + \left(-\ddot{\theta}\frac{1}{\sin\theta} + \dot{\phi}^2\cos\theta\right)\hat{z} - \left(\ddot{\phi} + 2\dot{\phi}\dot{\theta}\frac{\cos\theta}{\sin\theta}\right)\hat{1}\times\hat{z}$$

Then (3) gives,

$$-I\omega_1\dot{\phi} + (I+ma^2)\frac{a+b}{a}\left(-\ddot{\theta}\frac{1}{\sin\theta} + \dot{\phi}^2\cos\theta\right) + mga = 0 \qquad (4)$$

$$I\omega_1\dot{\theta} - (I+ma^2)\frac{a+b}{a}\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right) = 0 \qquad (5)$$

If $\dot{\theta} = 0$ and $\dot{\phi} = \Omega$, (4) reduces to

$$(I+ma^2)\frac{a+b}{a}\cos\theta\,\Omega^2 - I\omega_1\Omega + mga = 0$$

whereas (5) vanishes identically. The condition for the existance of the real solutions is

$$I^2\omega_1^2 - 4\cdot mga\cdot(I+ma^2)\frac{a+b}{a}\cos\theta \geq 0$$

$$\Rightarrow \omega_1 \geq \frac{2}{I}\sqrt{mg(a+b)(I+ma^2)\cos\theta}$$

The normal component of force exerted by the sphere can be calculated as follows.

$$\hat{1}\cdot\vec{F} = \hat{1}\cdot\left(mg\hat{z} + m\frac{d}{dt}(\vec{a}\times\vec{\omega})\right) = \hat{1}\cdot\left(mg\hat{z} + m(a+b)\frac{d^2}{dt^2}\hat{1}\right)$$

$$= mg\cos\theta + m(a+b)(-\Omega^2 + \Omega^2\cos^2\theta) \qquad (\text{for } \dot{\theta} = 0,\ \dot{\phi} = \Omega)$$

Thus, $\hat{1}\cdot\vec{F}\geq 0$ is equivalent to

$$\Omega^2 \leq \frac{g\cos\theta}{(a+b)\sin^2\theta}$$

Now we consider the nutation about the steady precession. We set

$$\theta = \theta_0 + \epsilon\sin\alpha t\ , \quad \dot{\phi} = \Omega + \delta\sin\alpha t$$

and we introduce

$$p = (I+ma^2)\,\frac{a+b}{a}\cdot\frac{1}{I}\ , \quad q = mga/I$$

for notational convenience. Assuming the nutation amplitude is small, we retain from (4) & (5) zeroth order equation and two first order equations in amplitudes (ϵ and δ). Then,

$$-\omega_1\Omega + p\Omega^2\cos\theta_0 + q = 0 \qquad (6)$$

$$\{-\omega_1\delta + p(2\Omega\delta\cos\theta_0 - \Omega^2\sin\theta_0\,\epsilon + \alpha^2\epsilon/\sin\theta_0)\}\sin\alpha t = 0 \qquad (7)$$

$$\{\omega_1\alpha\epsilon - p(\alpha\delta\sin\theta_0 + 2\Omega\epsilon\alpha\cos\theta_0)\}\cos\alpha t = 0 \qquad (8)$$

We write equations (7) and (8) in matrix form.

$$\begin{pmatrix} 2\Omega p\cos\theta_0 - \omega_1 & -p\left(\Omega^2\sin\theta_0 - \frac{\alpha^2}{\sin\theta_0}\right) \\ -p\sin\theta_0 & -(2\Omega p\cos\theta_0 - \omega_1) \end{pmatrix}\begin{pmatrix}\delta \\ \epsilon\end{pmatrix} = 0$$

Clearly the determinant should vanish to give nontrivial nutation. Thus,

$$\begin{aligned} p^2(\Omega^2\sin^2\theta_0 - \alpha^2) &= -(2\Omega p\cos\theta_0 - \omega_1)^2 \\ &= -\omega_1^2 + 4\omega_1\Omega p\cos\theta_0 - 4\Omega^2p^2\cos^2\theta_0 \\ &= -\omega_1^2 + 4\omega_1\Omega p\cos\theta_0 - 4p\cos\theta_0(\omega_1\Omega - q) = -\omega_1^2 + 4p\cos\theta_0\,q \end{aligned}$$

where we used (6) in right-~~ahan~~hand-side of the above calculations. Thus,

$$\alpha^2 = \frac{\omega_1^2 - 4p\cos\theta_0\,q}{p^2} + \Omega^2\sin^2\theta_0$$

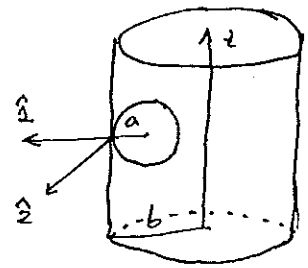
which is same as

$$\alpha^2 = \frac{I^2\omega_1^2 - 4mg(a+b)(I+ma^2)\cos\theta_0}{\left[(I+ma^2)\left(\frac{a+b}{a}\right)\right]^2} + \Omega^2\sin^2\theta_0$$

2. The Golfer's Nemesis

The force equation and torque equation are

$$\vec{F} - mg\hat{z} = m\frac{d}{dt}\vec{v}$$
$$I\frac{d}{dt}\vec{\omega} = \vec{a}\times\vec{F}$$



$\vec{a} = a\hat{1}$

respectively. By combining these two equations, we get,

$$I\frac{d}{dt}\vec{\omega} = \vec{a}\times(mg\hat{z} + m\frac{d}{dt}\vec{v})$$

The angular velocity vector can be written as,

$$\vec{\omega} = \frac{\dot{z}}{a}\hat{2} + \omega_1\hat{1} + \Omega\hat{z}$$

where the first term represents the rolling without slipping. By the direct differentiation, we get

$$\frac{d}{dt}\vec{\omega} = \frac{\ddot{z}}{a}\hat{2} + \dot{\omega}_1\hat{1} + \dot{\Omega}\hat{z} - \frac{\dot{z}}{a}\Omega\hat{1} + \omega_1\Omega\hat{2}$$

where we used

$$\frac{d}{dt}\hat{1} = \Omega\hat{2}, \quad \frac{d}{dt}\hat{2} = -\Omega\hat{1}, \quad \frac{d}{dt}\hat{z} = 0.$$

Since the velocity vector can be written as,

$$\vec{v} = \dot{z}\hat{z} + b\Omega\hat{2}$$

$\frac{d}{dt}\vec{v}$ is given by,

$$\frac{d}{dt}\vec{v} = \ddot{z}\hat{z} - b\Omega^2\hat{1}$$

which yields,

$$m\vec{a}\times\frac{d}{dt}\vec{v} = -ma\ddot{z}\hat{2}$$

Plugging these equations into the combined equation, we get

$$\hat{1}: \quad \dot{\omega}_1 = \dot{z}/a\ \Omega \qquad (1)$$
$$\hat{2}: \quad I\frac{\ddot{z}}{a} + ma\ddot{z} = -mga - I\omega_1\Omega \qquad (2)$$
$$\hat{z}: \quad \dot{\Omega} = 0 \Rightarrow \Omega: \text{constant}$$

(1) can be directly integrated to give

$$\omega_1 = \frac{z}{a} + \omega_{10}$$

where we used the given boundary conditions at t=0. Putting this into (2), we get,

$$(I+ma^2)\ddot{z} = -mga^2 - Ia\omega_{10}\Omega - I\Omega^2 z$$

$$\ddot{z} + \omega_z^2 z = -\frac{mga^2 + I\omega_{10}a\Omega}{I+ma^2} \qquad \left(\omega_z^2 \equiv \Omega^2\frac{I}{I+ma^2}\right)$$

The solution of this equation can be written as,

$$z = A\cos\omega_z t + B\sin\omega_z t - \frac{1}{\omega_z^2}\cdot\frac{mga^2 + I\omega_{10}a\Omega}{I+ma^2}$$

The initial condition $\dot{z}=0$ dictates $B=0$ and the condition $z=0$ gives,

$$A = \frac{1}{\omega_z^2}\cdot\frac{mga^2 + I\omega_{10}a\Omega}{I+ma^2} = \frac{mga^2 + I\omega_{10}a\Omega}{I\Omega^2}$$

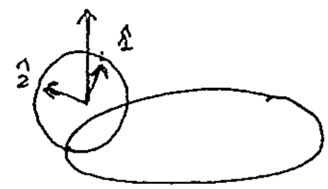
$$\therefore z = \frac{ma^2g + I\omega_{10}a\Omega}{I\Omega^2}(\cos\omega_z t - 1)$$

For sphere, $I = \frac{2}{5}ma^2$. Thus,

$$\frac{\Omega}{\omega_z} = \frac{2\pi n}{2\pi} = n = \sqrt{\frac{I+ma^2}{I}} = \sqrt{\frac{7}{2}} \approx 1.87$$

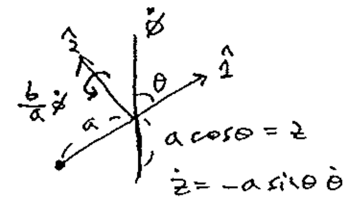
which means the ball rises again to the rim after 1.87 revolution.

3. Off the Rim



Since there is no component 1 of angular velocity vector by assumption, the angular velocity vector can be written as,

$$\vec{\omega} = \dot{\phi}\sin\theta\,\hat{2} - \frac{b}{a}\dot{\phi}\,\hat{2} + \dot{\theta}\,\hat{3}$$



where the first term represents the projection of "angular rotation along the rim" into axis 2 and the second term represents the rolling without slipping rotation and the third term, rotation falling into the basket. Thus, the rotational kinetic energy can be written as,

$$L_1 = \frac{1}{2}I\left((\dot{\phi}\sin\theta - \frac{b}{a}\dot{\phi})^2 + \dot{\theta}^2\right)$$

The kinetic energy of CM motion is clearly given by

$$L_2 = \frac{1}{2}m(a^2\sin^2\theta\,\dot{\theta}^2 + (b-a\sin\theta)^2\dot{\phi}^2)$$

with the potential energy

$$V = mga\cos\theta$$

where we used a constraint,

$$z = a\cos\theta$$

Thus, the total Lagrangian can be written as,

$$L = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\dot{\theta}^2 + \frac{1}{2}I(b-a\sin\theta)^2\frac{\dot{\phi}^2}{a^2} + \frac{1}{2}m(b-a\sin\theta)^2\dot{\phi}^2 - mga\cos\theta$$

which asserts that the effective potential is,

$$V_{eff}(\theta) = m\left\{ga\cos\theta - \frac{1}{2}\frac{I+ma^2}{ma^2}(b-a\sin\theta)^2\dot{\phi}^2\right\}$$

By the direct differentiation, we get, $(I = \frac{2}{3}ma^2$, in this case$)$

$$\frac{\partial}{\partial\theta}V_{eff} = 0 \Rightarrow \quad -ga\sin\theta + \frac{5}{3}(b-a\sin\theta)\cdot a\cos\theta\,\dot{\phi}^2 = 0$$

$$\dot{\phi}^2 = \frac{3g\tan\theta}{5(b-a\sin\theta)}$$

Further differentiation shows that

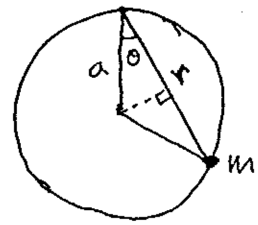
$$\frac{\partial^2}{\partial\theta^2}V_{eff} = m\left(-ga\cos\theta + \frac{5}{3}a\dot{\phi}^2(-\sin\theta)(b-a\sin\theta) + \frac{5}{3}a\dot{\phi}^2(-a\cos\theta)\cdot\cos\theta\right) < 0$$

$$(\because \cos\theta > 1 \quad \text{for} \quad 0 \le \theta \le \pi/2$$

That means our equilibrium is unstable. Intuitively, if $\Omega > \Omega_{eq.}$ then the ball will leave the hoop abd if $\Omega < \Omega_{eq.}$, the ball will fall into the basket.

4. (a) From the right figure, we see that

$$r = 2a\cos\theta$$



The Lagrangian can be written as as,

$$L = \frac{1}{2}m\left(r^2(\dot\theta+\Omega)^2 + \dot r^2\right)$$

$$= 2ma^2\left(\dot\theta^2 + 2\cos^2\theta\,\dot\theta\,\Omega + \cos^2\theta\,\Omega^2\right)$$

since there is no additional potential. One thing which simplifies the calculation is that the second term (which corresponds to Coriolis force term) is a total derivative of time. Thus, in calculation of the equation of motion, we can neglect the term. Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow 2\frac{d}{dt}\dot\theta - \frac{\partial}{\partial\theta}(\cos^2\theta\,\Omega^2) = 2\ddot\theta + 2\cos\theta\sin\theta\,\Omega^2 = 0 \Rightarrow \ddot\theta = -\Omega^2\sin\theta\cos\theta$$

(b) The Hamiltonian is given by

$$H = \dot\theta\frac{\partial L}{\partial\dot\theta} - L = \left\{\dot\theta(2\dot\theta + 2\Omega\cos^2\theta) - (\dot\theta^2 + 2\cos^2\theta\,\dot\theta\,\Omega + \cos^2\theta\,\Omega^2)\right\}2ma^2 = 2ma^2(\dot\theta^2 - \cos^2\theta\,\Omega^2)$$

where we should replace $\dot\theta$ in terms of the canonical momentum

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = 2ma^2(2\dot\theta + 2\Omega\cos^2\theta) = 4ma^2(\dot\theta + \cos^2\theta\,\Omega)$$

We put (3) into (2) and get,

$$H = 2ma^2\left(\left(\frac{p_\theta}{4ma^2} - \Omega\cos^2\theta\right)^2 - \Omega^2\cos^2\theta\right) \qquad \text{cf. } \left(\Omega^2(\cos^4\theta - \cos^2\theta) = \Omega^2\cos^2\theta\sin^2\theta = \frac{\Omega^2}{4}\sin^2 2\theta\right)$$

$$= \frac{p_\theta^2}{8ma^2} - p_\theta\,\Omega\cos^2\theta - \frac{ma^2}{2}\sin^2 2\theta$$

Since there is no explicit time dependence, we have

$$\frac{d}{dt}H = 0 \quad\Rightarrow\quad \ddot\theta = -\Omega^2\sin\theta\cos\theta$$

Also, the Hamilton's equation gives,

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} \Rightarrow \frac{d}{dt}\{4ma^2(\dot\theta + \cos^2\theta\,\Omega)\} = -2\cos\theta\sin\theta\,\Omega\,p_\theta + 2ma^2\sin 2\theta\cos 2\theta\cdot 2\dot\theta$$

$$\Rightarrow \ddot\theta = -\Omega^2\sin\theta\cos\theta$$

(c) In rotating coordinate system, there are Coriolis force and centrifu~~l~~gal force. Since the motion is purely angular, the centrifugal force should be radial which would be compensated by the constraint force. The same fact holds for the radial component of centrifugal force. Thus, retaining only the tangential component of the centrifugal force gives,

$$ma\,\ddot\alpha = -mr\Omega^2\sin\theta$$

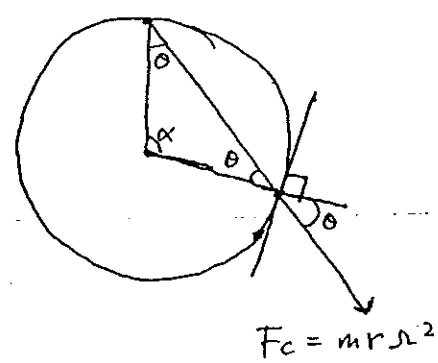
(See the right figure) Since,

$$\alpha = 2\theta \qquad (r = 2a\cos\theta)$$

this equation reduces to

$$\ddot\theta = -\sin\theta\cos\theta\,\Omega^2$$

as before.



5. The Piano

From the note (p.230), we have the formula

$$S(x,t) = \sum_n \phi_n \sin\frac{n\pi}{\ell}x$$

$$\phi_n(t) = \frac{\ell}{\rho\cdot\pi n c}\int_{-\infty}^{t} F_n(r)\sin\frac{n\pi c}{\ell}(t-r)\,dr$$

$$F_n(t) = \frac{2}{\ell}\int_0^{\ell} F(x,t)\sin\frac{n\pi}{\ell}x\,dx$$

Since our force is given by

$$F(x,t) = \begin{cases} F\delta(x-b)\sin 2\pi t/T & 0<t<\frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

F_n would be given by

$$F_n = \begin{cases} \frac{2}{\ell}F\sin\frac{n\pi b}{\ell}\sin\frac{2\pi}{T}t & 0<t<T/2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for the time t > T/2 , we have

$$\phi_n(t) = \frac{2F}{\rho\cdot\pi n c}\int_0^{T/2}\sin\frac{n\pi b}{\ell}\sin\frac{2\pi r}{T}\sin\frac{n\pi c}{\ell}(t-r)\,dr$$

$$= \frac{FT}{\rho\pi^2 n c}\sin\frac{n\pi b}{\ell}\int_0^{\pi}\sin\phi\sin(\phi_0-\alpha\phi)\,d\phi$$

where we introduced the rescaled parameters,

$$\frac{2\pi r}{T} = \phi, \quad \frac{n\pi c t}{\ell} = \phi_0, \quad \frac{nCT}{2\ell} = \alpha$$

Since,

$$\int_0^{\pi}\sin\phi\sin(\phi_0-\alpha\phi)\,d\phi = \frac{1}{2}\int_0^{\pi}\{\cos(\phi_0-(1+\alpha)\phi) - \cos(\phi_0+(1-\alpha)\phi)\}\,d\phi$$

$$= \frac{1}{2}\left(-\frac{\sin(\phi_0-(1+\alpha)\pi)-\sin\phi_0}{1+\alpha} - \frac{\sin(\phi_0+(1-\alpha)\pi)-\sin\phi_0}{1-\alpha}\right)$$

$$= \frac{1}{1-\alpha^2}\left(\sin(\phi_0-\alpha\pi)+\sin\phi_0\right)$$

$$\cancel{= \frac{2}{1-\alpha^2}(\sin\phi_0+\sin(\phi_0-\alpha\pi))} = \frac{2}{1-\alpha^2}\cos\left(\frac{\alpha\pi}{2}\right)\sin\left(\phi_0-\frac{\alpha\pi}{2}\right)$$

ϕ_n becomes

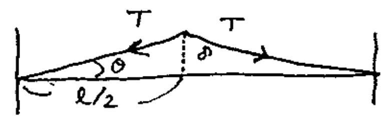
$$\phi_n = \frac{2FT}{\rho\pi^2 n c}\sin\frac{n\pi b}{\ell}\cdot\frac{1}{1-\alpha^2}\cos\left(\frac{\alpha\pi}{2}\right)\sin\left(\phi_0-\frac{\alpha\pi}{2}\right) \Rightarrow S(x,t) = \frac{2FT}{\pi^2 c\rho}\sum_n\frac{\sin\frac{n\pi b}{\ell}\cos\frac{n\pi cT}{4\ell}}{n\left(1-\left(\frac{ncT}{2\ell}\right)^2\right)}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi c}{\ell}\left(t-\frac{T}{4}\right)$$

If we choose b = 1/2 , then the above equation becomes, ($c^2 = T'/\rho$, T' : tension)

$$S(x,t) = \frac{2F\ell}{\pi^2 T'}\sum_n\frac{\sin n\pi}{n(1-n^2)}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi c}{\ell}\left(t-\frac{\ell}{2c}\right)$$

The above formula gives no contribution except for the case when n=1.

6. (a) We assume that the tension is approximately the constant, as usual. Then, the configuration shown to the right indicates the equation of motion,



$$M\frac{d^2}{dt^2}\delta = 2T\sin\theta$$

Since the angle θ is very small, we can approximate this as,

$$\sin\theta \sim \tan\theta = \frac{\delta}{\ell/2} = \frac{2}{\ell}\delta$$

Thus, the frequency is

$$\frac{d^2}{dt^2}\delta = \frac{4T}{M\ell}\delta \Rightarrow \Omega^2 = 2\sqrt{\frac{T}{M\ell}}$$

(b) As derived in the notes, the wave equation we should solve is, (for $x \neq b$)

$$\frac{\partial^2}{\partial t^2}y_1 = \frac{T}{\rho}\frac{\partial^2}{\partial x^2}y_1 \quad 0 \le x \le b, \quad \frac{\partial^2}{\partial t^2}y_2 = \frac{T}{\rho}\frac{\partial^2}{\partial x^2}y_2, \quad b \le x \le \ell$$

We have boundary conditions,

$$y_1(0,t) = y_2(\ell,t) = 0 \qquad (1)$$

$$y_1(b,t) = y_2(b,t) \qquad (2)$$

and

$$M\frac{\partial^2}{\partial t^2}y_1(b) = T\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right)\Big|_{x=b} \qquad (3)$$

where we use y_1 to denote the waves in the left, and y_2 in the right. The solutions of the above equation satisfying the conditions (1) are

$$y_1 = A\sin kx\cos(\omega t + \phi_0)$$
$$y_2 = B\sin k(\ell - x)\cos(\omega t + \phi_0) \qquad \left(\text{with } k^2 = \frac{\omega^2}{c^2},\ c^2 = T/\rho\right) \qquad (4)$$

where the phase in time variable should be the same for the boundary condition (2) to hold at all times. Now (2) gives,

$$A\sin kb = B\sin k(\ell - b)$$

and (3) gives,

$$M\frac{1}{y_1}\frac{\partial^2}{\partial t^2}y_1 = T\left(\frac{1}{y_2}\frac{\partial y_2}{\partial x} - \frac{1}{y_1}\frac{\partial y_1}{\partial x}\right)\Big|_{x=b} \Rightarrow \frac{Mc}{T}\omega = \frac{\cos k(\ell-b)}{\sin k(\ell-b)} + \frac{\cos k\ell}{\sin k\ell}$$

which can be rearranged to yield, $(\Omega = \omega)$

$$\frac{T}{Mc}\sin k\ell = \Omega\sin k\ell\sin k(\ell-b) \Rightarrow \frac{T}{Mc}\sin\frac{\Omega}{c}\ell = \Omega\sin\frac{\Omega b}{c}\sin\frac{\Omega}{c}(\ell-b)$$

(c) We rewrite the above condition as, $(b = \ell/2)$

$$\frac{T}{Mc}\sin\frac{\Omega}{c}\ell = \frac{2T}{Mc}\sin\frac{\Omega\ell}{2c}\cos\frac{\Omega\ell}{2c} = \Omega\sin\left(\frac{\Omega\ell}{2c}\right)\cdot\sin\frac{\Omega\ell}{2c}$$

which shows that there can be two possible cases. Either

i. $\sin\frac{\Omega\ell}{2c} = 0$. This condition reduces to

$$\frac{\Omega\ell}{2c} = n\pi \Rightarrow \Omega = \frac{2n\pi c}{\ell}$$

and M does not move as shown in (4).

ii. The remaining condition can be reduced into

$$\frac{\Omega\ell}{2c}\tan\left(\frac{\Omega\ell}{2c}\right) = \frac{\ell}{2c}\cdot\frac{2T}{Mc} = \frac{\rho\ell}{M} = \frac{m}{M}$$

and in this case M does move. $\left(\because \sin\frac{\Omega\ell}{2c} \neq 0\right)$

(d) We consider the second condition. By writing,

$$\beta = \frac{\Omega \ell}{2c}, \quad \alpha = \frac{m}{M}$$

our condition reduces to

$$\beta \tan\beta = \alpha$$

If α is very large, it is natural to assume,

$$\beta = \frac{\pi}{2} + \Delta\beta$$

and also assume $\Delta\beta \ll \frac{\pi}{2}$. Then above equation becomes,

$$\frac{1}{\alpha} = \frac{1}{\beta \tan\beta} = \frac{\cos(\frac{\pi}{2}+\Delta\beta)}{(\pi/2+\Delta\beta)\sin(\frac{\pi}{2}+\Delta\beta)} \approx \frac{-\Delta\beta}{\pi/2}, \quad \Delta\beta = -\frac{\pi}{2}\cdot\frac{1}{\alpha} = -\frac{\pi}{2}\cdot\frac{M}{m}.$$

Thus,

$$\beta = \frac{\pi}{2}\left(1-\frac{M}{m}\right) \Rightarrow \Omega = \frac{\pi c}{\ell}\left(1-\frac{M}{m}\right) = \Omega_1\left(1-\frac{M}{m}\right)$$

(e) If α is very small, β should be also small. By expanding, $\tan\beta \simeq \beta + \frac{\beta^3}{3}$, we have

$$\beta\left(\beta + \frac{1}{3}\beta^3\right) \approx \alpha$$

As a first approximation, we set,

$$\beta^2 = \alpha$$

Then the linear correction can be written as $\beta = \sqrt{\alpha}+\Delta\beta$. We put this into the equation and get

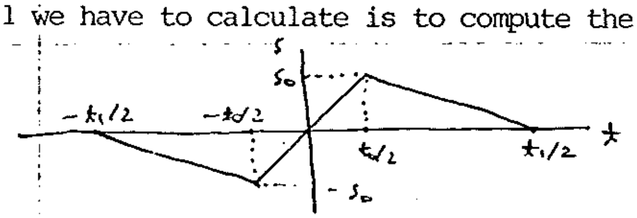
$$(\sqrt{\alpha}+\Delta\beta)\left(\sqrt{\alpha}+\Delta\beta + \frac{1}{3}(\sqrt{\alpha}+\Delta\beta)^3\right) \approx \left(\sqrt{\alpha}+\Delta\beta + \frac{1}{3}\alpha^{3/2}\right)(\sqrt{\alpha}+\Delta\beta)$$
$$\approx \alpha + 2\sqrt{\alpha}\Delta\beta + \frac{1}{3}\alpha^2 = \alpha$$

Thus, (cf. $\Omega_0 = 2\sqrt{\frac{T}{M\ell}} = 2c\cdot\sqrt{\frac{\rho}{M\ell}} = 2\frac{c}{\ell}\sqrt{\alpha}$)

$$\Delta\beta = -\frac{1}{6}\alpha^{3/2}, \quad \therefore \beta = \sqrt{\alpha}\left(1-\frac{1}{6}\alpha\right) \Rightarrow \Omega = \frac{2c}{\ell}\sqrt{\alpha}\left(1-\frac{1}{6}\alpha\right) = \Omega_0\left(1-\frac{m}{6M}\right)$$

7. Violin

All we have to calculate is to compute the Fourier expansion of the function,



This function can be written as,

$$s = \begin{cases} 2s_0\, t/t_0 & |t| < t_0/2 \\ s_0\dfrac{t_1-2t}{t_1-t_0} & \dfrac{t_0}{2} < t < \dfrac{t_1}{2} \\ s_0\dfrac{2t-t_1}{t_1-t_0} & -\dfrac{t_1}{2} < t < -\dfrac{t_0}{2} \end{cases}$$

Clearly, since the function is ~~even~~ odd function, there can not be any co sine terms.
Additionally, there is no constant term since the average of the above function is zero.
The ~~co~~sine-term-coefficients are given by, (since $s(t)$ is odd)

$$a_n = \frac{4}{t_1}\left(\int_0^{\frac{t_0}{2}} 2s_0 t/t_0 \sin\frac{2\pi n}{t_1}t\,dt + \int_{t_0/2}^{t_1/2} s_0\frac{t_1-2t}{t_1-t_0}\sin\frac{2\pi n}{t_1}t\,dt\right)$$

(integration by parts)

$$= \frac{4}{t_1}\cdot\frac{t_1}{2\pi n}\left(\frac{2s_0}{t_0}\int_0^{\frac{t_0}{2}}\cos\frac{2\pi n}{t_1}t\,dt + \frac{-2s_0}{t_1-t_0}\int_{t_0/2}^{t_1/2}\cos\frac{2\pi n}{t_1}t\,dt\right)$$

$$= \frac{4}{t_1} \cdot \left(\frac{t_1}{2\pi n}\right)^2 \cdot 2S_0 \left(\frac{1}{t_0} + \frac{1}{t_1 - t_0}\right) \sin\frac{2\pi n}{t_1} \cdot \frac{t_0}{2}$$

$$= \frac{2 S_0 t_1^2}{n^2 \pi^2 t_0 (t_1 - t_0)} \cdot \sin\frac{n\pi t_0}{t_1}$$

Using $\frac{x_0}{\ell} = \frac{t_0}{t_1}$, we have

$$S(t) = \frac{2 S_0 t_1^2}{\pi^2 t_0 (t_1 - t_0)} \sum_n \frac{1}{n^2} \sin n\pi \frac{x_0}{\ell} \sin\frac{2n\pi t}{t_1}$$

8. From the Young's law, the force at a certain point of spring is given by

$$F = k\ell_0 \frac{dS}{dx}$$

Thus, the equation of motion becomes,

$$dx\rho \frac{d^2}{dt^2} S = F(x+dx) - F(x) = k\ell_0 \frac{d^2}{dx^2} S \cdot dx$$

$$\frac{m}{k\ell_0^2} \frac{d^2}{dt^2} S = \frac{d^2}{dx^2} S \qquad (1)$$

where $\rho = m/\ell_0$, the density of the bar which is assumed to be a constant. The solution of the above equation can be written as,

$$S = A_0 \sin \Omega x \cos(\omega t + \phi_0) \quad (2)$$

at $x=0$.

where we satisfy the boundary condition $S=0$ ✓. The dispersion relation we get from (1) is

$$\Omega^2 = \frac{m\omega^2}{k\ell_0^2}$$

The boundary condition at the end where the mass M is attached is given by

$$M \ddot{S}(x=\ell_0) = F(x=\ell_0) = k\ell_0 \frac{d}{dx} S(x=\ell_0)$$

We put (2) into this equation and get,

$$M\omega^2 \sin\Omega\ell_0 = k\ell_0 \Omega \cos\Omega\ell_0$$

$$\Rightarrow \frac{M}{m}(\Omega\ell_0) = \cot\Omega\ell_0$$

If we write $\Omega\ell_0 = \beta$ and $\frac{m}{M} = \alpha$, this equation becomes

$$\beta \tan\beta = \alpha$$

which is identical to the equation considered at problem 6.(e). By quoting the result, we have, (for $\alpha \ll 1$)

$$\beta = \sqrt{\alpha} \cdot \left(1 - \frac{1}{6}\alpha\right) \approx \sqrt{\frac{\alpha}{1 + \frac{1}{3}\alpha}} \Rightarrow \sqrt{\frac{m}{k}}\,\omega = \sqrt{\frac{m}{M + m/3}}$$

$$\omega = \sqrt{\frac{k}{M + m/3}}$$