1. We have:

$$T_{\text{rot}} = \frac{1}{2}I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 \quad .$$

Where the θ, ϕ, ψ are Euler angles.

$$T_{\text{cm}} = \frac{1}{2}ma^2\dot\theta^2\cos^2\theta$$

and

$$V = mag\sin\theta$$

Therefore:

$$L = \frac{1}{8}ma^2(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{4}ma^2(\dot\phi\cos\theta + \dot\psi)^2 + \frac{1}{2}ma^2\dot\theta^2\cos^2\theta - mga\sin\theta$$

From the Lagrange, we have

$$\frac{d}{dt}(\dot\phi\cos\theta + \dot\psi) = 0 \tag{1}$$

$$\frac{d}{dt}(\dot\phi\sin^2\theta + 2(\dot\phi\cos\theta + \dot\psi)\cos\theta) = 0 \tag{2}$$

$$\frac{ma^2}{4}\frac{d}{dt}((1 + 4\cos^2\theta)\dot\theta) = -ma^2\cos\theta\sin\theta\dot\theta\dot\theta + \frac{1}{4}ma^2\dot\phi^2\cos\theta\sin\theta - \frac{1}{2}ma^2(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta - mga\cos\theta \tag{3}$$

For steady motion $\theta =$ const. and $\dot\phi =$ const. $\dot\psi =$ const. From (3) we have:

$$\frac{1}{4}\dot\phi^2\sin\theta\cos\theta - \dot\phi(\dot\phi\cos\theta + \dot\psi)\sin\theta - \frac{g}{a}\cos\theta = 0$$

Define:

$$\omega_1 \equiv \dot\phi\cos\theta + \dot\psi = \quad \text{angular momentum about symmetry axis}$$

So:

$$\frac{1}{4}\dot\phi^2\sin\theta\cos\theta - \frac{1}{2}\dot\phi\omega_1\sin\theta - \frac{g}{a}\cos\theta = 0$$

Solve the equation we have:

$$\dot\phi = \frac{\omega_1 \pm \sqrt{\omega_1^2 + \frac{4g\cos^2\theta}{a\sin\theta}}}{\cos\theta}$$

For $\theta = \frac{\pi}{2}, \omega_1 = 0, \dot{\phi}$ is arbitrary. From (1) we have:

$$\dot{\omega}_1 = 0 \quad \Rightarrow \quad \omega_1 = 0 \quad \text{always}$$

Let's define $\theta = \frac{\pi}{2} + \epsilon$, from (2) we have:

$$\dot{\phi}\sin^2\theta = \dot{\phi}_0 = \quad \text{constant}$$

therefore, $\dot{\phi} = \dot{\phi}_0$ to first order in ϵ. Now, expand (3) to first order in ϵ we have

$$\cos(\frac{\pi}{2} + \epsilon) \approx -\epsilon \quad ; \quad \sin(\frac{\pi}{2} + \epsilon) \approx 1 \quad ; \quad \dot{\theta} \approx \dot{\epsilon}$$

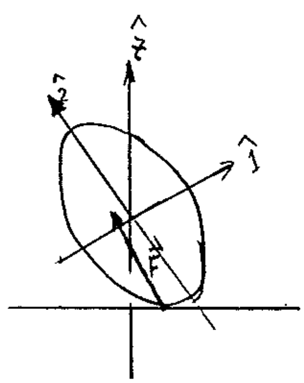
So:

$$\frac{ma^2}{4}\ddot{\epsilon} \approx -\frac{1}{4}ma^2\dot{\phi}^2\epsilon + mga\epsilon$$
$$\ddot{\epsilon} \approx \left(\frac{4g}{a} - \dot{\phi}^2\right)\epsilon$$

Therefore, the orbit is stable if

$$\dot{\phi} > 2\sqrt{\frac{g}{a}}$$

2.



$$\hat{z} = \cos\theta\hat{1} + \sin\theta\hat{2} \tag{1}$$

The angular velocity of the $\hat{1}, \hat{2}, \hat{3}$ frame is

$$\begin{aligned}\vec{\omega}_a =& \dot{\phi}\hat{z} - \dot{\theta}\hat{3} \\ =& \dot{\phi}\cos\theta\hat{1} + \dot{\phi}\sin\theta\hat{2} - \dot{\theta}\hat{3}\end{aligned} \tag{2}$$

The total angular velocity of the object is:

$$\vec{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{1} + \dot{\phi}\sin\theta\hat{2} - \dot{\theta}\hat{3}$$

and $\omega_1 = \dot{\psi} + \dot{\phi}\cos\theta$.

The angular velocity of the the object is:

$$\begin{aligned}\vec{L} = I\vec{\omega} =& (I_1\omega_1\hat{1} + I_2\dot{\phi}\sin\theta\hat{2} - I_3\dot{\theta}\hat{3}) \\ =& \frac{ma^2}{4}(2\omega_1\hat{1} + \dot{\phi}\sin\theta\hat{2} - \dot{\theta}\hat{3})\end{aligned} \tag{3}$$

From the condition of rolling without slipping, we have

$$\begin{aligned}\vec{v} =& \vec{a}\times\vec{\omega} = -a\hat{2}\times\vec{\omega} \\ =& a(\omega_1\hat{3} + \dot{\theta}\hat{1})\end{aligned} \tag{4}$$

From Newton's law and (4) we have:

$$\begin{aligned}\vec{F}_{\text{tot}} =& \vec{F}_{\text{contact}} - mg\hat{z} = ma\frac{d}{dt}(\omega_1\hat{3}\dot{\theta}\hat{1}) \\ \vec{F}_{\text{contact}} =& mg\hat{z} + ma\frac{d}{dt}(\omega_1\hat{3} + \dot{\theta}\hat{1})\end{aligned} \tag{5}$$

And

$$\frac{d\vec{L}}{dt} = -a\hat{2}\times\vec{F}_{\text{contact}}$$

From (5) and (3) we have:

$$\frac{1}{4}\frac{d}{dt}(2\omega_1\hat{1} + \dot{\phi}\sin\theta\hat{2} - \dot{\theta}\hat{3}) = \frac{g}{a}\cos\theta\hat{3} - \hat{2}\times\frac{d}{dt}(\dot{\theta}\hat{1} + \omega_1\hat{3}) \tag{6}$$

This is the equation of motion. If $\omega_1, \dot{\phi}$ and θ are constants, then (6) becomes:

$$\frac{1}{4}\left(2\omega_1\frac{d}{dt}\hat{1} + \dot{\phi}\sin\theta\frac{d}{dt}\hat{2}\right) = \frac{g}{a}\cos\theta\hat{3} - (\hat{2}\times\frac{d\hat{3}}{dt})\omega_1 \tag{7}$$

equation (2) becomes $\vec{\omega}_a = \dot{\phi}\hat{z}$ we have:

$$\begin{aligned}\frac{d\hat{1}}{dt} =& \omega_a\times\hat{1} = -\dot{\phi}\sin\theta\hat{3} \\ \frac{d\hat{2}}{dt} =& \omega_a\times\hat{2} = \dot{\phi}\cos\theta\hat{3} \\ \frac{d\hat{3}}{dt} =& -\dot{\phi}\cos\theta\hat{2} + \dot{\phi}\sin\theta\hat{1}\end{aligned}$$

From (7) we have:

$$\dot{\phi}^2\sin\theta\cos\theta - 6\omega_1\dot{\phi}\sin\theta - \frac{4g}{a}\cos\theta = 0$$

For $\theta = \frac{\pi}{2}$, we have the rolling solution $\dot{\phi} = 0$. Oscillations about $\theta = \frac{\pi}{2}$ and $\dot{\phi} = 0$. Let's set $\theta = \frac{\pi}{2} + \epsilon$ with $\epsilon << 1$ and $\dot{\phi} << \sqrt{\frac{g}{a}}$. From (6), to first order in ϵ and $\dot{\phi}$ we have

$$\begin{aligned}&\frac{1}{2}\dot{\omega}_1\hat{1} + \frac{1}{2}\omega_1\frac{d}{dt}\hat{1} + \frac{1}{4}(\dot{\phi}\hat{2} - \dot{\theta}\hat{3}) + \frac{1}{4}\vec{\omega}_a \times (\dot{\phi}\hat{2} - \dot{\theta}\hat{3}) \\ &= -\frac{g}{a}\epsilon\hat{3} - (\hat{2}\times\hat{1})\ddot{\theta} - \hat{2}\times(\vec{\omega}_a \times (\dot{\theta}\hat{1} + \omega_1\hat{3})) - (\hat{2}\times\hat{3})\dot{\omega}_1\end{aligned} \tag{8}$$

To fist order in ϵ and $\dot{\phi}$, $\vec{\omega}_a \approx \dot{\phi}\hat{2} - \dot{\theta}\hat{3}$. So, from (8) we have:

$$\frac{3}{2}\dot{\omega}_1\hat{1} + (\frac{1}{4}\dot{\phi} - \frac{1}{2}\omega_1\dot{\theta})\hat{2} - (\frac{1}{4}\ddot{\theta} + \frac{1}{4}\dot{\phi})\hat{2} = (\frac{g\epsilon}{a} + \ddot{\theta} + \omega_1\dot{\phi})\hat{3}$$

so:

$$\begin{aligned}\dot{\omega}_1 &= 0 \quad \Rightarrow \quad \omega_1 = \text{ const.} \\ \ddot{\phi} &= 2\omega_1\dot{\theta} \quad \Rightarrow \quad \dot{\phi} = 2\omega_1\epsilon \\ \frac{5}{4}\ddot{\theta} &= \frac{\epsilon g}{a} - \frac{3\omega_1}{2}\dot{\phi}\end{aligned} \tag{11}$$

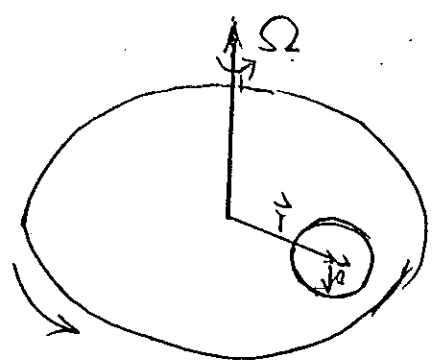
Using (11), we have

$$\frac{5}{4}\ddot{\epsilon} = -\epsilon(3\omega_1^2 - \frac{g}{a})$$

Therefore, stable condition is:

$$\omega_1^2 > \frac{g}{3a}$$

3.



(a) The condition of rolling without slipping is:

$$0 = \vec{v} + \vec{\omega} \times \vec{a} - \vec{\Omega} \times \vec{r}$$

Therefore:

$$\vec{v} = \vec{\Omega} \times \vec{r} - \vec{\omega} \times \vec{a}$$

(b)

$$\begin{aligned} \vec{N} &= \frac{d\vec{L}}{dt} \\ \vec{L} &= \mathbf{I} \cdot \vec{\omega} \end{aligned}$$

$$\vec{N} = \vec{a} \times \vec{F} = \frac{mv^2 a}{r}\hat{1}$$

where $v^2 = (\Omega r - \omega a)^2$ but

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\omega\frac{v}{r}\hat{1}$$

Therefore:

$$\frac{mv^2 a}{r} = \frac{Iv\omega}{r}$$

or

$$m(\Omega r - \omega a)a = I\omega$$

Therefore:

$$\omega = \frac{\Omega}{a}\left(\frac{ma^2 r}{I + ma^2}\right)$$

(c)

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{d}{dt}(\vec{\Omega} \times \vec{r} - \vec{\omega} \times \vec{a}) \\ &= \vec{\Omega} \times \frac{d\vec{r}}{dt} - \frac{d\vec{\omega}}{dt} \times \vec{a} \end{aligned} \qquad (1)$$

However:

$$\vec{F} = m\frac{d\vec{v}}{dt} \quad ; \quad \vec{N} = \vec{a} \times \vec{F}$$

Therefore:

$$\frac{d\vec{\omega}}{dt} = \frac{m\vec{a}}{I} \times \frac{d\vec{v}}{dt}$$

So, (1) yields

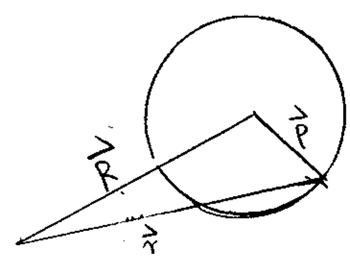
$$\begin{aligned}\frac{d\vec{v}}{dt} =& \vec{\Omega} \times \frac{d\vec{r}}{dt} - (\frac{m\vec{a}}{I} \times \frac{d\vec{v}}{dt}) \times \vec{a} \\ =& \vec{\Omega} \times \vec{v} - \frac{ma^2}{I}\frac{d\vec{v}}{dt}\end{aligned}$$

So, we have:

$$\frac{d\vec{v}}{dt} = \frac{I}{I + ma^2}\vec{\Omega} \times \vec{v}$$

Integrating this:

$$\begin{aligned}\vec{v} =& \frac{I}{I + ma^2}\vec{\Omega} \times \int \vec{v}dt \\ =& \frac{I}{I + ma^2}(\vec{\Omega} \times \vec{\rho})\end{aligned}$$



as:

$$\vec{r} = \vec{R} + \vec{\rho}$$

we have

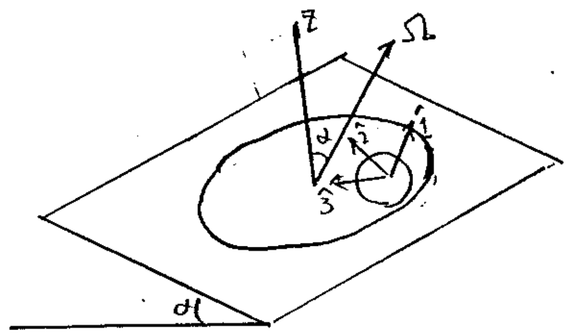
$$\vec{R} = \vec{r}_0 + \frac{I + ma^2}{I\Omega^2}\vec{\Omega} \times \vec{v}_0$$

$$\begin{aligned}\vec{\omega} \times \vec{a} =& \frac{I}{I + ma^2}\Omega \times (\vec{r} - \vec{R}) - \vec{\Omega} \times \vec{r} \\ =& \vec{\Omega} \times \left(\frac{ma^2}{I + ma^2}(-\vec{r}) + \frac{I}{I + ma^2}(-\vec{R})\right)\end{aligned}$$

But as $\vec{a} \perp \vec{\omega}$ we have

$$\vec{\omega} = \frac{\Omega}{a}\left(\frac{I\vec{R} + ma^2\vec{r}}{I + ma^2}\right)$$

(d)



$$\frac{d\vec{v}}{dt} = \vec{\Omega} \times \frac{d\vec{r}}{dt} + \vec{a} \times \frac{d\vec{\omega}}{dt}$$

F is the force provided by the table:

$$\vec{F} - mg\sin\alpha\hat{2} = m\frac{d\vec{v}}{dt}$$

and we have

$$\vec{N} = \vec{a} \times \vec{F}$$

we have

$$\frac{d\vec{\omega}}{dt} = \frac{m\vec{a}}{I} \times \frac{d\vec{v}}{dt} + \frac{mg\sin\alpha}{I}(-\hat{3})$$

Therefore:

$$\begin{aligned}\frac{d\vec{v}}{dt}\left(1 + \frac{ma^2}{I}\right) =& \vec{\Omega} \times \frac{d\vec{r}}{dt} - \frac{mga^2\sin\alpha}{I}\hat{2} \\ =& \vec{\Omega} \times \left(\frac{d\vec{r}}{dt} + \frac{mga^2\sin\alpha}{I\Omega}\hat{3}\right)\end{aligned}$$

Let:

$$\vec{v} = \vec{v}_e + \vec{v}_d$$

with

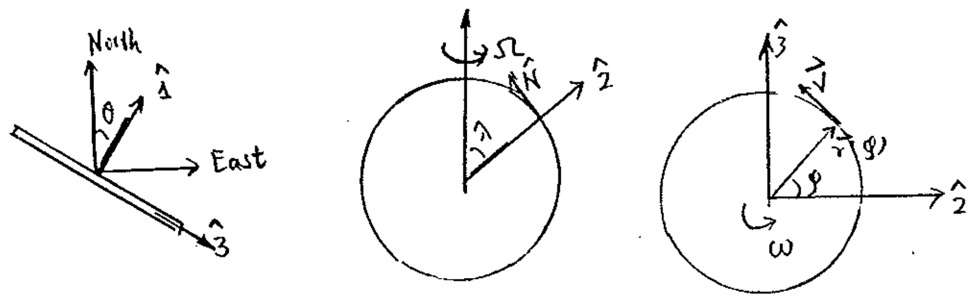
$$\vec{v}_d = -\frac{mga^2\sin\alpha}{\Omega I}\hat{3}$$

and:

$$\frac{d\vec{v}_e}{dt} = \frac{I}{I + ma^2}\vec{\Omega} \times \vec{v}_e$$

which we have already solved in part (c).

4.



(a) We have:

$$I_1 = Ma^2 \quad ; \quad I_2 = I_3 = \frac{1}{2}Ma^2$$

and

$$\vec{\Omega} = \Omega(\cos\lambda\hat{2} + \sin\lambda\hat{N})$$
$$\vec{N} = \cos\theta\hat{1} - \sin\theta\hat{2}$$

So

$$\vec{\Omega} = \Omega(\sin\lambda\cos\theta\hat{1} + \cos\theta\hat{2} - \sin\theta\sin\lambda\hat{3})$$

$$d\vec{N} = \vec{r}\times(-2dm\vec{\Omega}\times\vec{v})$$

with

$$dm = \frac{M}{2\pi}d\phi \quad ; \quad \vec{v} = \vec{\omega}\times\vec{r}$$
$$\vec{r}(\phi) = a\cos\phi\hat{2} + a\sin\phi\hat{3}$$
$$\vec{\omega} = o\hat{1}$$

$$\vec{r}\times(\vec{\Omega}\times\vec{v}) = (\vec{r}\cdot\vec{v})\vec{\Omega} - (\vec{r}\cdot\vec{\Omega})\vec{v}$$
$$= a^2\Omega\omega(\cos\lambda\cos\phi - \sin\lambda\sin\phi\sin\theta)(\sin\phi\hat{2} - \cos\phi\hat{3})$$

Therefore:

$$\vec{N} = \int_0^{2\pi} d\vec{N}(\phi)$$
$$= Ma^2\omega\Omega(\sin\lambda\sin\theta\hat{2} + \cos\lambda\hat{3})$$

Now, we note that $\omega_2 = -\dot{\theta}$ because $\hat{2}$ is up, so we have:

$$-\ddot{\theta}I_2 = N_2 = Ma^2\omega\Omega\sin\lambda\sin\theta \approx Ma^2\omega\Omega\sin\lambda\theta$$

As

$$I_2 = \frac{1}{2}Ma^2$$

we have:

$$\ddot{\theta} \approx -2\omega\Omega \sin\lambda\theta$$

Therefore, it gives oscillatory solution with frequency $\sqrt{2\omega\Omega \sin\lambda}$

(b) As before:

$$\begin{aligned}\vec{\omega} &=(\omega, 0, 0)\\ \vec{\dot{\theta}} &=(0, -\dot{\theta}, 0)\end{aligned}$$

and

$$\vec{\Omega} = \Omega(\sin\lambda\cos\theta, \cos\lambda, -\sin\lambda\sin\theta)$$

For a disk:

$$I_1 = \frac{1}{2}Ma^2 \quad ; \quad I_2 = I_3 = \frac{1}{4}Ma^2$$

So:

$$\begin{aligned}\vec{\omega}_{\text{tot}} &=\vec{\omega} + \vec{\dot{\theta}} + \vec{\Omega}\\ &=(\omega + \Omega\sin\lambda\cos\theta, \Omega\cos\lambda - \dot{\theta}, -\Omega\sin\lambda\sin\theta)\end{aligned}$$

So:

$$\vec{L} = I_2(2\omega + \Omega 2\sin\lambda\cos\theta, \Omega\cos\lambda - \dot{\theta}, -\Omega\sin\lambda\sin\theta)$$

And

$$\frac{dL^*}{dt} = I_2(2\dot{\omega} - \Omega 2\sin\lambda\sin\theta\dot{\theta}, -\ddot{\theta}, -\Omega\sin\lambda\cos\theta\dot{\theta})$$

$$\begin{aligned}&(\vec{\dot{\theta}} + \vec{\Omega}) \times \vec{L}\\ =&I_2\begin{vmatrix}\hat{1} & \hat{2} & \hat{3}\\ \Omega\sin\lambda\cos\theta & \Omega\cos\lambda - \dot{\theta} & -\Omega\sin\lambda\sin\theta\\ 2\omega + 2\Omega\sin\lambda\cos\theta & \Omega\cos\lambda - \dot{\theta} & -\Omega\sin\lambda\sin\theta\end{vmatrix}\\ =&-I_2(2\omega + \Omega\sin\lambda\cos\theta)\Omega\sin\lambda\sin\theta\hat{2} - I_2(2\omega + \Omega\sin\lambda\cos\theta)(\Omega\cos\lambda - \dot{\theta})\hat{3}\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{dL^*}{dt} + (\vec{\dot{\theta}} + \vec{\Omega}) \times \vec{L} = I_2(2\dot{\omega} - \Omega 2\sin\lambda\sin\theta\dot{\theta}, -\ddot{\theta} - (2\omega + \Omega\sin\lambda\cos\theta)\Omega\sin\lambda\sin\theta,\\ -\Omega^2\sin\lambda\cos\theta\dot{\theta} - (2\omega + \Omega\sin\lambda\cos\theta)(\Omega\cos\lambda - \dot{\theta})) = 0\end{aligned}$$

From the $\hat{1}$ term:

$$2\dot{\omega} = 2\Omega\sin\lambda\sin\theta\dot{\theta}$$

But, $\dot{\theta}$ and θ are small, so $\omega \approx$ const. The $\hat{2}$ term:

$$\ddot{\theta} = -2\omega\Omega\sin\lambda\sin\theta - \Omega^2\sin^2\lambda\cos\theta\sin\theta$$

But $\Omega << \omega$ and $\theta << 1$, we have

$$\ddot{\theta} \approx -2\omega\Omega \sin\lambda\theta$$

so, the angular frequency is:

$$\sqrt{2\omega\Omega\sin\lambda}$$

as in part (a).

5.

$$\begin{aligned} L^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \\ 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \end{aligned}$$

From pg. 196 of the note, we have:

$$0 = 2TI_2 - L^2 = I_1(I_2 - I_1)\omega_1^2 + I_3(I_2 - I_3)\omega_3^2$$

Thus:

$$\omega_1^2 = \frac{I_2 - I_3}{I_1 - I_2}\frac{I_3}{I_1}\omega_3^2$$

Then:

$$\begin{aligned} L^2 - I_2^2\omega_2^2 &= (I_1^2\frac{I_2 - I_3}{I_1 - I_2}\frac{I_3}{I_1} + I_3^2)\omega_3^2 \\ &= \frac{I_3 I_2(I_1 - I_3)}{I_1 - I_2}\omega_3^2 \end{aligned}$$

Now ω_1 and ω_3 are symmetric, so:

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1 I_2} = \frac{I_2 - I_3}{I_1 - I_3}\frac{L^2 - I_2\omega_2^2}{I_1 I_2}$$

From Euler equation with $\vec{N} = 0$

$$0 = I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3)$$

Or:

$$\begin{aligned} \dot{\omega}_2 &= -\frac{1}{I_2}(L^2 - I_2^2\omega_2^2)\sqrt{\frac{(I_1 - I_2)(I_2 - I_3)}{(I_1 - I_3)^2 I_1 I_3 I_2^2}}(I_1 - I_3) \\ &= -((\frac{L}{I_2})^2 - \omega_2^2)\sqrt{\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}} \end{aligned}$$

Let:

$$k = \sqrt{\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}}$$

$$\omega_j\mathrm{max} = \frac{L}{I_j}$$

So:

$$\dot{\omega}_2 = -k\omega_{2\mathrm{max}}^2 + k\omega_2^2$$

So:

$$\omega_2 = \omega_{2\mathrm{max}} \tanh(k\omega_{2\mathrm{max}}(t + t_0))$$

And:

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{(I_2 - I_3)}{(I_1 - I_2)} \frac{I_2}{I_1}} \frac{(k\omega_{2\mathrm{max}}^2 - k\omega_2^2)^{1/2}}{\sqrt{k}} \\
&= \sqrt{\frac{(I_1 - I_3)I_2}{(I_1 - I_3)I_1}} \frac{L}{I_2} \mathrm{sech}(k\omega_{2\mathrm{max}}(t + t_0)) \\
&= \sqrt{\frac{(I_2 - I_3)I_1}{(I_1 - I_3)I_2}} \omega_{1\mathrm{max}} \mathrm{sech}(k\omega_{2\mathrm{max}}(t + t_0))
\end{aligned}$$

By symmetry, we have:

$$\omega_3 = \sqrt{\frac{(I_2 - I_1)I_3}{(I_3 - I_1)I_2}} \omega_{3\mathrm{max}} \mathrm{sech}(k\omega_{2\mathrm{max}}(t + t_0))$$