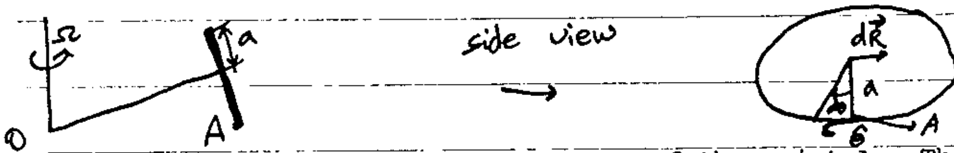
Ph 205 Problem Set 1 - Solutions

* For full scores for each problem, see page 12.

1. Let us first contemplate what we mean by "instantaneous axis". We consider the disk rotating about a fixed axis with contant Ω connected to the origin by the massless rod.



As to this motion, we can think of the movement of the point A. The location of A, say $\vec{r}$ can be written as two parts,

$$\vec{r} = \vec{R} + \vec{r}' \qquad (1)$$

where $\vec{R}$ represents the vector from O to center of the disk and $\vec{r}'$ is the vector from the center to A. Now, we diffrentiate (1) and get,

$$\dot{\vec{r}} = \frac{d\vec{R}}{dt} + \frac{d}{dt}\vec{r}' = \frac{d\vec{R}}{dt} + a\frac{d\theta}{dt}\hat{\theta} \qquad (2)$$

The condition of the rolling without slipping gives,

$$dR = a\,d\theta \quad \text{and} \quad \hat{\theta} = -\hat{R} \qquad (3)$$

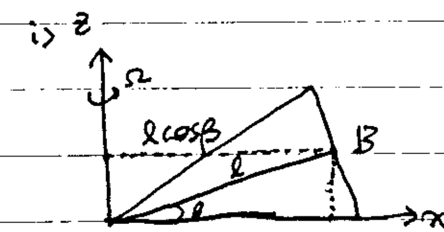
where $\vec{R}=R\hat{R}$, since th distance moved by the disk should be provided by the rolling. By (3) we get,

$$\dot{\vec{r}} = 0$$

That means point A momentarily stops. Since a cone has basically the same kinematics, we can immediately see that the points contacting ground form "instantaneous axis", i.e., we can momentarily view the motion as the rotation about this momentarily fixed axis.

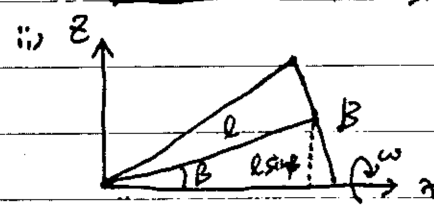
Now, we have two methods of viewing the same motion. The velocity $\vec{v}$ of the point B can be obtained either by i) considering the rotation about the given axis z, i.e.,

$$\vec{v} = \text{radius} \times \text{angular speed} = l\cos\beta\,\Omega\,\hat{y}$$



or by ii) considering the rotation about the momentary axis x, i.e.,

$$\vec{v} = \omega\, l \sin\beta\, \hat{y}$$

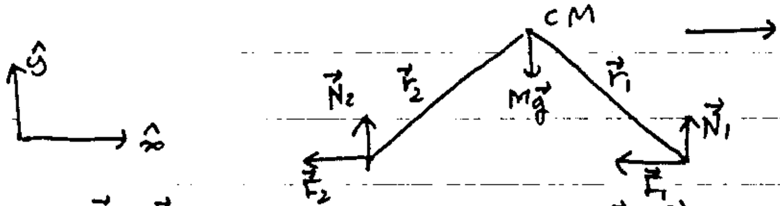


By equating two equations, we get

$$\omega l \sin\beta = l\cos\beta\,\Omega$$

$$\therefore\ \omega = \Omega\frac{\cos\beta}{\sin\beta} = \Omega\cot\beta.$$

2. We assume a usual car has the reflection symmetry with respect to the central surface when viewed from front. Consequently, we can represent the car as shown below, where relevant forces are shown.



where $\vec{F}_1$, $\vec{F}_2$ are frictional forces and $\vec{N}_1$, $\vec{N}_2$ represent normal forces exerted by the road. Now, we have three equations, namely,

$$\vec{F}_1 + \vec{F}_2 = M\vec{a} \quad \text{; Newton's law along the road} \qquad (1)$$

$$\vec{N}_1 + \vec{N}_2 + M\vec{g} = 0 \quad \text{; Newton's law vertical to the road} \qquad (2)$$

$$\frac{d}{dt}\vec{L} = \vec{r}_1 \times \vec{N}_1 + \vec{r}_2 \times \vec{N}_2 + \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = 0 \qquad (3)$$

; Angular momentum equation if we regard C.M. as a reference point. In (3), we used the fact that $\vec{L}$ vanishes since the car does not rotate with respect to CM.

The first task we should do is to find the minimum value of $\vec{a}$ $(a = a\hat{x})$. (Notice that $a<0$ in my convention). Rewriting (1), (2) and (3), we have,

$$Ma = -F_1 - F_2$$
$$N_1 + N_2 = Mg$$
$$l_1 N_1 - l_2 N_2 - h(F_1 + F_2) = 0 \qquad \text{c.f. } \vec{F}_1 = -F_1\hat{x},\ \vec{N}_1 = N_1\hat{y},\ etc)$$

Thus,

$$Ma = -\frac{1}{h}(l_1 N_1 - l_2 N_2) = -\frac{l_1}{h}Mg + \frac{l_1 + l_2}{h}N_2 \qquad (4)$$

Theoretically, although we are given the sume of N_1 and N_2, we can freely change the ratio of N_1 and N_2 by changing the efficiency of the rear and front brakes (e.g. we can use the diffrence between rolling friction coefficient and slipping friction coefficient). Thus, we can interpret N_2 as an independent varable having the range of

$$0 \le N_2 < Mg \qquad (5)$$

Notice that only positive N_2 is physically relevant. Thus, the maximum braking is achieved when $N_2 = 0$, with the $Ma = -\frac{l}{h}Mg$. The reason why we need a good brake on the front wheels is that we need decelerating effect to come dominantly from the front wheels to get maximum deceleration. As long as deceleration is concerned, in which $F_1 + F_2$ should be positive, the condition under which the car would leave the road is $N_2 = 0$, (not $N_1 = 0$) since the car needs no support from the ground. In this case, (4) gives,

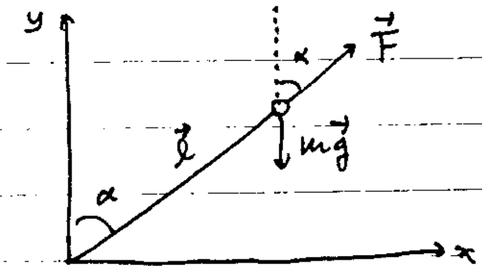
$$F_1 + F_2 = \frac{l_1}{h}Mg = Ma \qquad (6)$$

If we assume the simple model of frictional force, namely, $F_{friction} = \mu F_{normal}$, (6) becomes

$$\mu = \frac{l_1}{h} \qquad (7)$$

since $N_2 = 0$ and $N_1 = Mg$.

3. The forces given to the mass are



Using the angular momentum equation, ($\vec{l} \parallel \vec{F}$)

$$\frac{d\vec{L}}{dt} = -ml\ddot{\alpha}\,\hat{z} = \vec{l}\times\vec{F} + \vec{l}\times(m\vec{g}) = -mgl\sin\alpha\,\hat{z}$$

and the Newton's law,

$$\vec{F} + m\vec{g} = m\ddot{\vec{l}} \Rightarrow \vec{F} = -m\vec{g} + m\ddot{\vec{l}} = -m\vec{g} + m(-l\dot{\alpha}^2\,\hat{l} + l\ddot{\alpha}\,\hat{\alpha})$$

we have

$$\ddot{\alpha} = \frac{g}{l}\sin\alpha \qquad (1)$$

$$\vec{F} = F\,\hat{l} \quad \text{with} \quad F = mg\cos\alpha - l\dot{\alpha}^2 \qquad (2)$$

since

$$-m\vec{g} = mg\cos\alpha\,\hat{l} - mg\sin\alpha\,\hat{\alpha}$$

Multiplying by $d\alpha$ and integrating yields, (from (1))

$$\frac{1}{2}\dot{\alpha}^2 - \frac{1}{2}\dot{\alpha}_0^2 = \frac{1}{2}\dot{\alpha}^2 = m\frac{g}{l}(\cos\alpha_0 - \cos\alpha) \qquad (3)$$

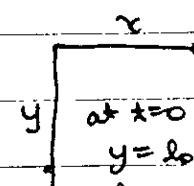
where we used the fact $\dot{\alpha}_0 = 0$.

Using (3),

$$F = mg(3\cos\alpha - 2\cos\alpha_0)$$

Initially F>0 (compression) and as α gets larger, F<0 (tension). The critical angle is given by

$$F = 0, \Rightarrow \cos\alpha = \frac{2}{3}\cos\alpha_0.$$



4. (a) The total mechanical energy of the system can be written as,

$$E = \frac{1}{2}\frac{m}{l}y\dot{y}^2 + \frac{1}{2}\frac{m}{l}x\dot{x}^2 - \frac{m}{l}y\cdot\frac{y}{2}$$

since since we regard the cable in two portions which represent y and x. In this case the total energy should be conserved. Since x+y=l, $\dot{x} = -\dot{y}$, we can write E as

$$E = \frac{1}{2}\frac{m}{l}y\dot{y}^2 + \frac{1}{2}\frac{m}{l}(l-y)\dot{y}^2 - \frac{1}{2}\frac{m}{l}gy^2$$

$$= \frac{1}{2}m\dot{y}^2 - \frac{1}{2}\frac{mg}{l}y^2$$

By taking time derivative, we get,

$$\frac{d}{dt}E = m\dot{y}\ddot{y} - \frac{g}{\ell}y\dot{y} = 0 \Rightarrow \ddot{y} = \frac{g}{\ell}y$$

The solution of above equation is

$$y = a\exp\left(\sqrt{\frac{g}{\ell}}t\right) + b\exp\left(-\sqrt{\frac{g}{\ell}}t\right)$$

Using two initial conditions, $\dot{y}(0)=0$ and $y(0)=l$, we get

$$a-b=0,\quad a+b=\ell \Rightarrow a=b=\frac{\ell}{2}$$

$$\therefore\ y = \frac{\ell}{2}\left(\exp\left(\sqrt{\frac{g}{\ell}}t\right) + \exp\left(-\sqrt{\frac{g}{\ell}}t\right)\right)$$

(b) The total momentum of the cable is

$$P = \frac{m}{\ell}y\dot{y}$$

Since the only external force which can contributes is gravittion force $F_{ext} = \left(\frac{m}{\ell}y\right)g$, we can use the formula

$$P_{tot} = \frac{d}{dt}\left(\frac{m}{\ell}y\dot{y}\right) = F_{ext} = \frac{m}{\ell}yg \qquad \therefore\ \frac{d}{dt}(y\dot{y}) = gy$$

Multiplying both sides by ydy and integrating gives,

$$y\dot{y}\,d(y\dot{y}) = gy^2dy$$

$$\frac{1}{2}y^2\dot{y}^2 - \frac{1}{2}y^2(0)\dot{y}(0)^2 = \frac{1}{3}g(y^3 - y_0^3)$$

Since $\dot{y}(0)=0$ and $y(0)=l_0$, we have,

$$\dot{y} = \sqrt{\frac{2}{3}gy\left(1-\frac{\ell_0^3}{y^3}\right)}$$

The total energy of the system is

$$E = \frac{1}{2}\left(\frac{m}{\ell}\right)y\dot{y}^2 - g\left(\frac{m}{\ell}y\right)\left(\frac{y}{2}\right)$$

$$= \frac{1}{3}\frac{mg}{\ell}y^2\left(1-\frac{\ell_0^3}{y^3}\right) - \frac{g}{2}\frac{m}{\ell}y^2$$

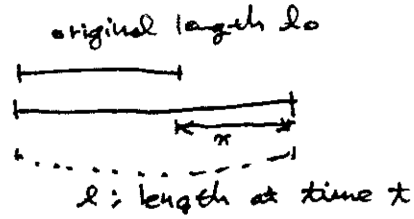
$$= -\frac{mg}{6\ell}\left(y^2 + \frac{2\ell_0^3}{y}\right) \qquad \therefore\ \frac{d}{dt}E = -\frac{mg}{3\ell}\left(y - \frac{\ell_0^3}{y^2}\right)\frac{dy}{dt}$$

In fact, since $\frac{dy}{dt} > 0$ and $y > \ell_0$, $\frac{dE}{dt} < 0$. The simple interpretation of this decreasing of energy is obtained by using

$$\dot{y}^2 = \frac{2}{3}g\left(y - \frac{\ell_0^3}{y^2}\right)$$

$$\therefore\ \frac{dE}{dt} = -\frac{1}{2}\frac{m}{\ell}\dot{y}^2\cdot\frac{dy}{dt} = -\frac{1}{2}\left(\frac{m}{\ell}\frac{dy}{dt}\right)\dot{y}^2 = -\frac{1}{2}\frac{dm}{dt}\dot{y}^2$$

dm above is the bit of the cable which would participat in the motion during dt. Initially it was at rest and suddenly got kinetic energy $\frac{1}{2}dm\,\dot{y}^2$ during dt. Since there's no source for the kinetic energy, it must be originated from the total mechanical energy of the system. Thus the total energy is decreased by the rate specified above. In conclusion, the missing energy has gone to supply the kinetic energy of the new-comer (dm) by an inelastic process.



5. We calculate the total energy of the system

$$E = \int_0^{l} \frac{1}{2}\left(\frac{m}{l}\right) dl' \cdot \left(\frac{l'}{l}\dot{l}\right)^2 + \frac{1}{2}kx^2 + \frac{1}{2}M\dot{l}^2$$

where $\frac{m}{l}$ is the line density of the spring and the velocity of the spring element at l' is $\frac{l'}{l}\dot{l}$ since the lefthand side of the spring is fixed and the righthand side is moving with the velocity of $\dot{l}$. Now,

$$E = \frac{1}{6}m\dot{l}^2 + \frac{1}{2}M\dot{l}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}\left(M + \frac{m}{3}\right)\dot{x}^2 + \frac{1}{2}kx^2 \qquad (1)$$

since $\dot{l} = \frac{d}{dt}(l_0 + x) = \dot{x}$

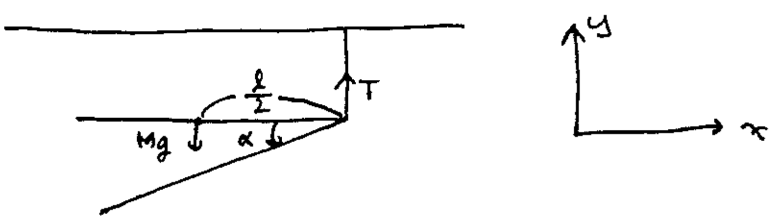
The energy of the simple harmonic oscillator is given by

$$E' = \frac{1}{2}m'\dot{x}^2 + \frac{1}{2}k'x^2 \qquad (2)$$

with E' conserved. The period of (2) is, $\tau' = 2\pi\sqrt{\frac{m'}{k'}}$. In our case, energy is conserved and by analogy,

$$\tau = 2\pi\sqrt{\frac{M + m/3}{k}}$$

6.



First, we consider the torque regarding O as a reference point. Then,

$$I\ddot{\alpha} = \frac{l}{2}Mg \qquad \text{at } t=0,\ \alpha = 0,\ \dot{\alpha} = 0 \qquad (1)$$

where α is the angle the rod makes with the horizontal line, $\frac{l}{2}$ denotes the distance to CM and I is the moment of inertia, which is calculated to be,

$$I = M\frac{\int_0^{l} x^2 dx}{\int_0^{l} dx} = \frac{1}{3}Ml^2 \qquad (2)$$

for the uniform rod. From the Newton's law,

$$T - Mg = m\ddot{y}$$

Since we can regard the motion of the rod as a rotation about O momentarily,

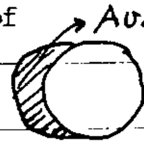
$$\ddot{y} = -\frac{l}{2}\ddot{\alpha}$$

Consequently,

$$T = Mg + M\ddot{y} = Mg - \frac{Ml}{2}\ddot{\alpha} = Mg - \frac{3}{4}Mg = \frac{1}{4}Mg.$$

where (1) and (2) has been used. Notice that this calculation is valid only momentarily.

7. (a) If v>>s, we can assume that air molecules are frozen, i.e. static. Thus during dt the sphere sweeps the volume of $Av\Delta t$

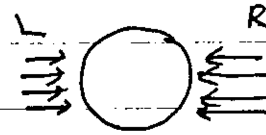


Thus, $\Delta m = \rho A v \Delta t$ molecules get momentum $v\Delta m$ from the sphere which means the drag force

$$F_{drag} = \frac{v\Delta m}{\Delta t} = \rho A v^2$$

(b) If v<<s, the amount of the mass of the air molecules which collides with the sphere during dt is

$$(\rho A s \Delta t) \times 50\% = \frac{\rho A s \Delta t}{2}$$



from L→R and the same amount from R→L. In former case, the gain in momentum is

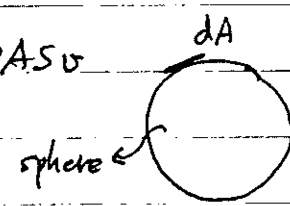
$$\frac{1}{2}\rho A s \Delta t \; (v+s) \qquad \xrightarrow[\text{initial}]{s} \quad \xleftarrow[\text{final}]{v}$$

while, in the latter case, the gain in momentum is

$$-\frac{1}{2}\rho A s \Delta t \; (v-s) \qquad \xleftarrow[\text{initial}]{s} \quad \xleftarrow[\text{final}]{v}$$

Thus, the drag force is

$$F_{drag} = \left\{ \frac{1}{2}\rho A s \Delta t \, (v-s) + \frac{1}{2}\rho A s \Delta t \, (v+s) \right\} / \Delta t = \rho A s v$$



(c) Consider a frame in which the sphere looks static. Then,

$$\vec{F} = \int_A \langle (\underbrace{\rho \vec{s}' \cdot \hat{n} \, dA \, dt}_{mass}) \vec{s}' \rangle_{average} \; / dt$$

where $\vec{F}$ is the drag force. The meaning of the above equation is that during dt, $-\rho \vec{s}' \cdot \hat{n} \, dA \, dt$ of airmoles stuck to the sphere, gaining $-\vec{s}'$ of momentum. Notice that in my convention $\hat{n}$ is the outward normal vector from the surface of the sphere. Thus, in the averaging process which is denoted as $\langle \;\; \rangle_{average}$ in the above equation, we should add only those contributions from the air molecules which has negative $\vec{s}' \cdot \hat{n}$. The molecules getting away from the sphere can not stick to it. In view of this fact, we can define the averaging process like this,

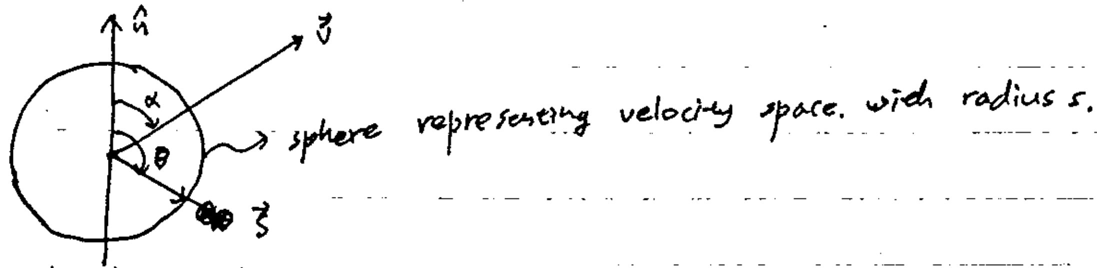
$$\langle \cdots \rangle_{average} = \frac{1}{4\pi s^2} \int s^2 \underbrace{d\Omega_s}_{solid\ angle} \times \begin{cases} \cdots\cdots & \text{if } \vec{s}' \cdot \hat{n} < 0 \\ 0 & \text{if } \vec{s}' \cdot \hat{n} > 0 \end{cases}$$

since $\vec{s}$ is randomly distributed. Notice also the diffrence between $\vec{s}'$ and $\vec{s}$, and the fact that the averaging integral appearing above is performed in "velocity space", rather than usual space.

In the lab frame, all we have to do is to change $\vec{s}' \to \vec{s} + \vec{v}$. Thus,

$$\vec{F} = \int_A \langle \rho (\vec{s} + \vec{v}) \cdot \hat{n} \, dA \; (\vec{s} + \vec{v}) \rangle_{average}$$

Carefully observe the diagram below in velocity space. ($s < v$)



The averaging integration introduced before should be evaluated at the circle spanned by all possible s, which is clearly a sphere. For a given α , we should add all the contribution from the sphere as long as,

$$(\vec{v}+\vec{s})\cdot\hat{n} \le 0 \Rightarrow v\cos\alpha + s\cos\theta \le 0 \Rightarrow \cos\theta \le -\frac{v}{s}\cos\alpha$$

Thus, if $-\frac{s}{v}\cos\alpha < -1 \Rightarrow \cos\alpha > \frac{s}{v}$, no θ contributes. If $-\frac{s}{v}\cos\alpha > 1 \Rightarrow \cos\alpha < -\frac{s}{v}$, ... (1)

all θ contributes. In this case, the averaging integral becomes

$$\vec{X}_{(\alpha)} = \frac{1}{4\pi}\int_0^{\pi}(\vec{s}+\vec{v})\cdot\hat{n}\,(\vec{s}+\vec{v})\,2\pi\sin\theta\, d\theta \quad \text{(due to the rotational symmetry about } \hat{n}\text{) \& s is constant}$$

$$= \frac{1}{2}\int_{-1}^{1}(\vec{s}+\vec{v})\cdot\hat{n}\,(\vec{s}+\vec{v})\,dx \quad (x=\cos\theta)$$

$$= \frac{1}{2}\int_{-1}^{1}(s\cos\theta+v\cos\alpha)(s\cos\theta\,\hat{n}+v\hat{v}) \quad \text{(due to the same rotational symmetry)}$$

$$= \frac{1}{2}\int_{-1}^{1}(s x+v\cos\alpha)(s x\hat{n}+v\hat{v})dx = \frac{1}{3}s^2\hat{n}+v^2\cos\alpha\,\hat{v}$$

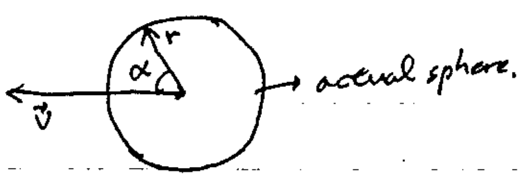
If $-\frac{s}{v} < \cos\alpha < \frac{s}{v}$, only $-1 < \cos\theta < -\frac{v}{s}\cos\alpha$ contributes. Thus, ... (2)

$$\vec{X}_{(\alpha)} = \frac{1}{2}\int_{-1}^{-\frac{v}{s}\cos\alpha}(sx+v\cos\alpha)(sx\hat{n}+v\hat{v})dx$$

$$= \frac{1}{2}\left(\frac{s^2}{3}\left(1-\frac{v^3}{s^3}\cos^3\alpha\right)+vs\frac{\cos\alpha}{2}\left(\frac{v^2}{s^2}\cos^2\alpha-1\right)\right)\hat{n}$$

$$+\frac{1}{2}\left(\frac{sv}{2}\left(\frac{v^2}{s^2}\cos^2\alpha-1\right)+v^2\cos\alpha\left(1-\frac{v}{s}\cos\alpha\right)\right)\hat{v}$$

Now we should perform the spatial integration on the sphere shown below



The straightforward calculation gives,

$$\vec{Y} = 2\pi\int_0^{\pi} r^2\sin\alpha\, d\alpha\, \vec{X}(\alpha) \quad \text{(due to rotational symmetry about } \vec{v}\text{)} \; (y=\cos\alpha)$$

$$= 2\pi r^2\left\{\int_{-1}^{-\frac{s}{v}}dy\left(\frac{1}{3}s^2\hat{n}+v^2y\hat{v}\right)+\frac{1}{2}\int_{-\frac{s}{v}}^{\frac{s}{v}}\left[\left(\frac{s^2}{3}\left(1-\frac{v^3}{s^3}y^3\right)+vs\frac{y}{2}\left(\frac{v^2}{s^2}y^2-1\right)\right)\hat{n} + \left(sv\frac{1}{2}\left(\frac{v^2}{s^2}y^2-1\right)+v^2y\left(1-\frac{v}{s}y\right)\right)\hat{v}\right]dy\right\}$$

(due to the axisymmetry)

$$= 2\pi r^2\hat{v}\left\{\int_{-1}^{-\frac{s}{v}}dy\left(\frac{1}{3}s^2+v^2\right)y+\frac{1}{2}\int_{-\frac{s}{v}}^{\frac{s}{v}}\left[\frac{s^2}{3}y\left(1-\frac{v^3}{s^3}y^3\right)+vs\frac{y^2}{2}\left(\frac{v^2}{s^2}y^2-1\right)+\frac{vs}{2}\left(\frac{v^2}{s^2}y^2-1\right) + v^2y\left(1-\frac{v}{s}y\right)\right]dy\right\}$$

$$(A=\pi r^2) \quad = -A\hat{v}\left(v^2+\frac{2}{3}s^2-\frac{1}{15}\frac{s^4}{v^2}\right)$$

Now the drag force can be written as,

$$\vec{F}_{drag} = \rho\vec{Y} = -\rho A\left(v^2+\frac{2}{3}s^2-\frac{1}{15}\frac{s^4}{v^2}\right)\hat{v}.$$

In case of s>v, the whole calculation is similar to those given above. In this case, all θ contributes for all α and the result of the velocity space averaging is given by, (just like case (2))

$$\vec{X}(\alpha) = \frac{1}{2}\left(\frac{s^2}{3}\left(1-\frac{v^3}{s^3}\cos^3\alpha\right) + vs\frac{\cos\alpha}{2}\left(\frac{v^2}{s^2}\cos^2\alpha-1\right)\right)\hat{n} + \frac{1}{2}\left(\frac{sv}{2}\left(\frac{v^2}{s^2}\cos^2\alpha-1\right)+v^2\cos\alpha\left(1-\frac{v}{s}\cos\alpha\right)\right)\hat{v}$$

Now we perform the spatial integration straightforwardly and get,

$$\vec{Y} = 2\pi\int_0^\pi r^2\sin\alpha\, d\alpha\, \vec{X}(\alpha) \qquad (x = \cos\alpha)$$

$$= 2\pi r^2\hat{v}\int_{-1}^{1}dx\cdot\frac{1}{2}\left\{\frac{s^2}{3}\left(1-\frac{v^3}{s^3}x^3\right)x + \frac{vs}{2}x^2\left(\frac{v^2}{s^2}x^2-1\right) + \frac{vs}{2}\left(\frac{v^2}{s^2}x^2-1\right)+v^2x\left(1-\frac{v}{s}x\right)\right\}$$

$$= -A\hat{v}\left(\frac{4}{3}vs + \frac{4}{15}\frac{v^3}{s}\right)$$

Thus in this case, the drag force is given by,

$$\vec{F}_{drag} = -\rho A\left(\frac{4}{3}vs + \frac{4}{15}\frac{v^3}{s}\right)\hat{v}$$

8. The governing equation in this case is given by,

$$m\frac{d}{dt}v + kr^2v = mg$$

By transforming $v=v' + \frac{mg}{kr^2}$, we get equation for v',

$$m\frac{d}{dt}v' = -kr^2v'$$

By the direct integration of the above solution we get,

$$v = \frac{mg}{kr^2} + C\exp\left(-\frac{kr^2}{m}t\right)$$

Using the boundary condition v=0 at t=0 we get the complete solution.

$$v = \frac{mg}{kr^2}\left(1-\exp\left(-\frac{kr^2}{m}t\right)\right)$$

By expanding exponential funtion using the Taylor series $e^x = 1+x+\frac{x^2}{2}+\cdots$, we get

$$v = gt\left(1-\frac{kr^2}{2m}t+\cdots\right)$$

Thus, $\epsilon = \frac{kr^2}{2m}$. This is simply the free falling somewhat draged by ϵ term. If t is sufficiently large, we can neglect the exponential term in our solution. Thus, the terminal velocity is given by,

$$v \approx \frac{mg}{kr^2} = v_{terminal}(r)$$

which is portional to inverse r square.

(b) By differentiating mass $m=\frac{4}{3}\pi r^3\rho$ with respect to time, we get

$$4\pi r^2\rho\frac{dr}{dt} = \alpha r^2 \Rightarrow \frac{d}{dt}r = \frac{\alpha}{4\pi\rho} \Rightarrow r = r_0 + \frac{\alpha}{4\pi\rho}t. \qquad (1)$$

In this case, our governing equation is given by,

$$\frac{d}{dt}(mv) + kr^2v = mg$$

We rewrite our in the form shown below.

$$\frac{d}{dt}\left(\rho r^3\frac{4}{3}\pi v\right) = +\frac{4}{3}\pi r^3\rho g - kr^2v \xrightarrow[\text{((1) is used)}]{} \frac{\alpha}{3}\frac{d}{dr}(r^3v) = +\frac{4}{3}\pi\rho g r^3 - kr^2v$$

Above form is equivalent to the form shown below which is directly integrable.

$$(\alpha r^2 + kr^2)v + \frac{\alpha}{3}r^3\frac{d}{dr}v = +\frac{4}{3}\pi\rho g r^3$$

$$\Rightarrow \quad \frac{\alpha}{3}r^{-3(\alpha+k)/\alpha}\frac{d}{dr}\left(r^{3(\alpha+k)/\alpha}v\right) = +\frac{4}{3}\pi\rho g$$

$$v r^{3(\alpha+k)/\alpha} - r_0^{3(\alpha+k)/\alpha} v_0 = + \frac{4\pi\rho g}{\alpha}\left(\frac{r^{3(\alpha+k)/\alpha+1}}{3(\alpha+k)/\alpha+1} - \frac{r_0^{3(\alpha+k)/\alpha+1}}{3(\alpha+k)/\alpha+1} \right)$$

Now, using $r = r_0 + \frac{\alpha}{4\pi\rho} t$,

$$v(t) = \frac{4\pi\rho g}{4\alpha+3k}\left(r_0 + \frac{\alpha}{4\pi\rho} t\right) + \left(v_0 - 4\pi\rho g \frac{r_0}{4\alpha+3k}\right) \frac{r_0^{3(\alpha+k)/\alpha}}{\left(r_0 + \frac{\alpha}{4\pi\rho} t\right)^{3(\alpha+k)/\alpha}}$$

In special case, $r_0 = v_0 = 0$, we get

$$v(t) = \frac{4\pi\rho g}{4\alpha+3k} r = \frac{g t}{4+3k/\alpha} = \frac{v_{terminal}(r)}{(4+3k/\alpha)\cdot\frac{\alpha}{3k}} = \frac{v_{terminal}(r)}{1+4\alpha/3k} < v_{terminal}(r)$$

$$\left(v_{terminal}(r) = \frac{mg}{kr^2} = \frac{4\pi}{3}\frac{\rho g}{k} r = \frac{g\alpha}{3k} t \right)$$

9. (a) A simple way to observe this system is to observe the cylinder moving with jerk. In this picture, the cylinder moves from the left and abruptly stops and falls over. From the moment it stops, we can visulaize motion as a rotation about a fixed point. We regard the point as a reference point measuring angular momentum. By doing this, although the translational motion has been abruptly stopped by the frictional impulse type force, the angular momentum should be continuous at the time the cylinder stops since the direction of the frictional force is ~~parrel~~ parallel to the direction from the reference point to the point of action, i.e., there is no impulse type torque. If this initial angular momentum was large enough to slant the cylinder as shown below, it will fall down.

fig. 1

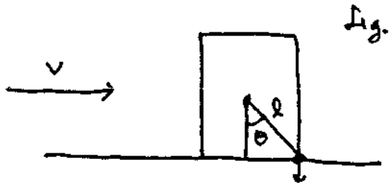
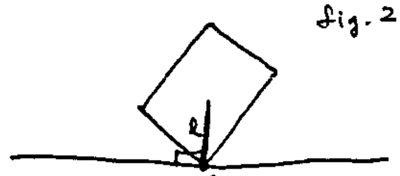


fig. 2



The initial angular momentum is easily calculated to be,

$$L_0 = |M \vec{l} \times \vec{v}| = Mlv\cos\theta$$

which can be equated with

$I\omega$; I → moment of inertia of cylinder with respect to reference point.

to yield the initial angular velocity ω, with respect to the refence point.

$$I\omega = Mlv\cos\theta, \Rightarrow \omega = \frac{Mlv\cos\theta}{I}$$

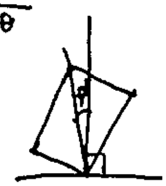
Using energy conservation, we get the condition, (see fig. 2)

$$\frac{1}{2}I\omega^2 + Mgl\cos\theta = \text{Rotational Energy (R.E.)} + Mgl, \quad \text{R.E.} \geq 0.$$

$$\frac{1}{2}I\omega^2 \geq Mgl(1-\cos\theta) \Rightarrow v^2 \geq \left(\frac{2I}{Ml^2}\right) \cdot lg\frac{1-\cos\theta}{\cos^2\theta}$$

The moment of inertia is calculated to be,

$$I = \frac{15+\cos^2\theta}{12} ml^2.$$



Thus, we have the desired result,

$$\therefore \quad v^2 \geq \frac{gl}{6}\frac{(15+\cos^2\theta)(1-\cos\theta)}{\cos^2\theta}$$

(b) The condition for the cylinder to fly off the ground is the normal force exerted by the ground vanishes. The Newton's law vertical to the plane is given by,

$$M\ddot{y} = T - mg \qquad (1)$$

where T is the normal force and $\ddot{y}$ is calculated as follows,

$$y = l\cos\phi, \quad \dot{y} = -l\sin\phi\,\dot{\phi}, \quad \ddot{y} = -l\cos\phi\,\dot{\phi}^2 - l\sin\phi\,\ddot{\phi}$$

Using the energy conservation,

$$\frac{1}{2}I\dot{\phi}^2 + Mgl\cos\phi = E_0$$

with the total energy E_0 = mgl (In the case when the equality of the (a) holds, the cylinder stops when the diagoanl line of the cylinder is vertial to the plane.

Thus, we have the total energy shown above.), we get

$$\dot{\phi}^2 = \frac{2}{I} Mg\ell (1-\cos\phi)$$

By differentiating the above result with respect to t, we get

$$2\dot{\phi}\ddot{\phi} = \frac{2}{I} Mg\ell \sin\phi \,\dot{\phi} \Rightarrow \ddot{\phi} = \frac{Mg\ell}{I}\sin\phi.$$

By putting these results into (1), we get the expression for Normal force,

$$T = Mg - M\ell\cos\phi \,\frac{2}{I} Mg\ell(1-\cos\phi) - M\ell\frac{Mg\ell}{I}\sin^2\theta$$
$$= Mg\cdot\frac{M\ell^2}{I}\left(\frac{I}{M\ell^2} - 2\cos\phi + 3\cos^2\phi - 1\right)$$

Consequently, the condition $T=0$ yields,

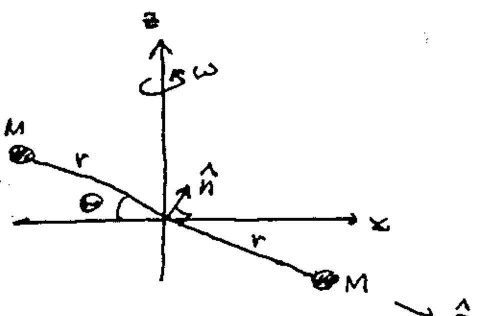
$$3\cos^2\phi - 2\cos\phi + \frac{3+\cos^2\theta}{12} = 0$$
$$\therefore \cos\phi = \frac{1}{3}\left(1 \pm \sqrt{1-\frac{3+\cos^2\theta}{4}}\right) = \frac{1}{3}\left(1 \pm \frac{1}{2}\sin\theta\right).$$

For the basement of the cylinder to leave the ground during its rise, the solution above should exists between $0 \le \phi \le \theta$. This gives us the result

$$\cos\theta \ge \frac{1}{3} + \frac{1}{6}\sin\theta.$$

cf. The moment of inertia can be calculated i) by direct integration, or ii) using parallel axis theorem & perpendicular axis theorem properly.

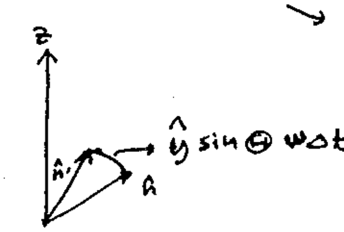
10. (a) Using the definition of $\vec{L}$,



$$\vec{L} = \sum_i m\vec{r}_i \times \vec{v}_i \qquad (\vec{r}_1 = r\hat{e}_1,\ \vec{v}_1 = r\cos\theta\,\omega(-\hat{y}))$$
$$(\vec{r}_2 = r\hat{e}_2,\ \vec{v}_2 = r\cos\theta\,\omega\hat{y})$$
$$= 2Mr^2\omega\cos\theta\,\hat{n}$$

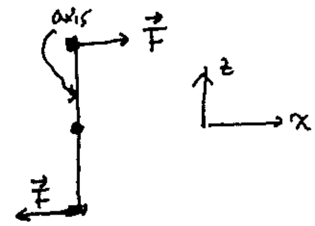
where $\hat{n}$ is shown in the right figure.

$$\frac{d\vec{L}}{dt} = \vec{N} = 2Mr^2\omega\cos\theta\frac{d}{dt}\hat{n} \quad \text{(see the right figure)}$$
$$= 2Mr^2\omega^2\cos\theta\sin\theta\,\hat{y}.$$



(b)

(b) If the required torque is supplied by two ~~both~~ bearings as shown, the direction of the force should be as shown.



The magnitude is

$$2F\cdot d = 2\cdot Mr^2\omega^2\cos\Theta\sin\Theta = N.$$

$$\therefore\ F = \frac{Mr^2\omega^2\cos\Theta\sin\Theta}{d}$$

(c) If the wheels ~~are~~ break free from the bearings, there's no external torque. At that time, the angular momentum should be continuous. Thus, the resulting motion is the rotation about $\hat{n}$ axis, with the magnitude of ω',

$$\underline{2Mr^2\omega'} = 2Mr^2\omega\cos\Theta, \Rightarrow \omega' = \omega\cos\Theta$$

Thus, the period of the motion is $\tau = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega\cos\Theta}$

(d) In spherical polar coordinate, one of the masses acceleration is given by,

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2)\hat{\theta} + (r\sin\theta\,\ddot{\phi} + 2\dot{r}\sin\theta\,\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi})\hat{\phi}$$

Since the mass is free to rotate as long as Θ is concerned, there's no force in $\hat{\theta}$ direction.

Thus $a_\theta = 0$. Moreover, $\theta = \frac{\pi}{2} - \Theta$, $\dot{\phi} = \omega$, $r = \text{constant}$. Consequently, we have

$$\ddot{\Theta} + \sin\Theta\cos\Theta\,\omega^2 = 0. \qquad (1)$$

If Θ is small, from (1), we approximately get

$$\ddot{\Theta} + \omega^2\Theta = 0.$$

Thus Θ oscillates about $\Theta = 0$, with the angular frequency ω.

Full Scores.

1. 10 2. 10 3. 10 4. (a) 5 (b) 5 5. 10

6. 10 7. (a) 2 (b) 2 (c) 6 8. (a) 5 (b) 5

9. (a) 5 (b) 5 10. (a) 3 (b) 2 (c) 2 (d) 3