PH 205 SET 12

DUE THURSDAY, DEC 20, 1990 ; MAXIMUM RECORDED SCORE = 80 POINTS

(1) RECALL PROBLEM (6), SET 11. A STRING OF MASS m, LENGTH ℓ HAS BOTH ENDS FIXED, AND A SMALL MASS M IS ATTACHED A DISTANCE b FROM ONE END. FIND THE SHIFT IN THE FREQUENCY OF THE nTH MODE USING RAYLEIGH'S PERTURBATION METHOD.

(2) A STRING OF LENGTH ℓ WITH BOTH ENDS FIXED HAS DENSITY

$$\rho(x) = \begin{cases} \rho_0 + \epsilon & 0 < x < b \\ \rho_0 - \epsilon & b < x < \ell \end{cases}$$

I.E. IT IS MADE OUT OF 2 STRINGS OF DIFFERENT DENSITY.

a) SOLVE THE WAVE EQUATION FOR EACH STRING SEPARATELY, AND MATCH SOLUTIONS AT $x = b$ TO SHOW

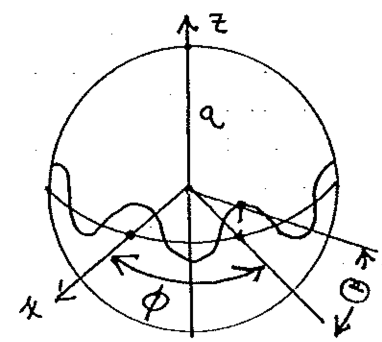
$$c_1 \tan \frac{\Omega b}{c_1} = - c_2 \tan \frac{\Omega(\ell - b)}{c_2} \qquad \text{WHERE } c_{1,2} = \sqrt{\frac{T}{\rho_0 \pm \epsilon}}$$

THIS IS EXACT, BUT NOT VERY TRANSPARENT.

b) USE RAYLEIGH'S PERTURBATION METHOD TO FIND THE SHIFT IN FREQUENCIES RELATIVE TO THE CASE $\epsilon = 0$. NOTE THAT IF $b = \ell/2$ THERE IS NO SHIFT — A SIMPLE RESULT NOT READILY APPARENT FROM PART a).

(3) A STRING IS STRETCHED AROUND THE EQUATOR OF A SPHERE OF RADIUS a. THE TENSION IS T, THE LINEAR DENSITY IS ρ.

LET $\Theta(\phi)$ = ANGULAR DISPLACEMENT OF THE STRING TRANSVERSE TO THE EQUATORIAL PLANE. USE LAGRANGE'S METHOD TO FIND THE WAVE EQUATION OF MOTION. BE CAREFUL ABOUT DIMENSIONS, AND REMEMBER THAT THE STRING ALWAYS LIES ON THE SURFACE OF THE SPHERE.



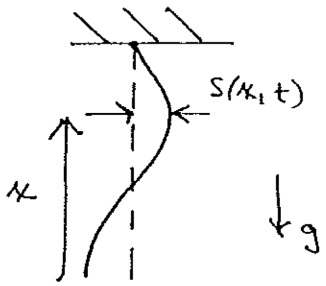
THE NORMAL MODES HAVE THE FORM $\Theta = \cos n\phi \cos \omega_n t$

SHOW $\omega_n^2 = \frac{T}{\rho a^2}(n^2 - 1)$

IF $n=0$ THE STRING WOULD POP OFF; IF $n=1$, IT IS NOT STRETCHED.

"PLANETARY STRING THEORY"

(4) AN INELASTIC STRING (CHAIN) OF MASS M, LENGTH ℓ HANGS VERTICALLY WITH ITS UPPER END FIXED.



a) FIND THE WAVE EQUATION FOR SMALL TRANSVERSE OSCILLATIONS. (IN A VERTICAL PLANE).

LET x BE MEASURED UPWARDS FROM THE BOTTOM END OF THE STRING. MAKE A CHANGE OF VARIABLE, $z = \sqrt{x}$ TO SHOW

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \frac{4\omega^2}{g} f = 0 \quad , \text{ SUPPOSING } S(x,t) = f(x)\cos\omega t.$$

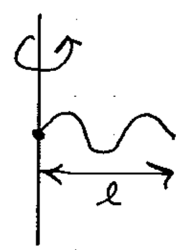
THIS IS KNOWN AS BESSEL'S EQUATION OF ORDER ZERO.

b) APPROXIMATE THE LOWEST FREQUENCY BY RAYLEIGH'S ENERGY METHOD. <u>DON'T</u> INCLUDE THE GRAVITATIONAL P.E. WHEN EQUATING $\langle K.E.\rangle = \langle P.E.\rangle$.

HINT: TRY $f(x) = \ell^P - x^P$. SHOW THAT $\omega = 1.207\sqrt{g/\ell}$

(COMPARED TO $\omega = 1.202\ldots\sqrt{g/\ell}$ EXACT)

(5) A STRING OF MASS M, LENGTH ℓ IS ATTACHED TO A SHAFT WHICH ROTATES RAPIDLY WITH CONSTANT ANGULAR VELOCITY Ω (IGNORE GRAVITY).



FIND THE EQUATION OF MOTION FOR <u>TRANSVERSE</u> VIBRATIONS IN THE PLANE CONTAINING THE SHAFT AND THE STRING.

MEASURE x OUTWARD FROM THE SHAFT. IF $z \equiv x/\ell$,

SHOW $$\frac{d}{dz}\left[(1-z^2)\frac{df}{dz}\right] + \frac{2\omega^2}{\Omega^2} f = 0 \quad \text{SUPPOSING } S(x,t) = f(x)\cos\omega t$$

THIS IS LEGENDRE'S EQUATION, MOSTLY SEEN IN SPHERICAL PROBLEMS (C.F. Ph 206). THE BOUNDARY CONDITION AT $x=0$ IS $f(0)=0$. AT $x=\ell$ ($\Rightarrow z=1$) IT IS SUFFICIENT TO REQUIRE THAT d^2f/dz^2 BE FINITE. THEN THE DIFFERENTIAL EQUATION REQUIRES

$$df/dz\big|_{z=1} = \frac{\omega^2}{\Omega^2} f(1) \quad \text{FOR ANY SOLUTION}$$

WHICH IS A KIND OF 'AUTOMATIC' BOUNDARY CONDITION. THE SOLUTIONS PROPOSED ON P 3 SATISFY THIS OF COURSE.

[SPECIALISTS NOTE THE EXISTENCE OF ANOTHER CLASS OF SOLUTIONS FOR WHICH $\frac{d^2f}{dz^2} \to \infty$ AS $z \to 1$. NAMELY $g_0 = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$; $g_1 = \frac{1}{2} z \ln\frac{z+1}{z-1} - 1 \ \ldots$]

THE SOLUTIONS TO LEGENDRE'S EQUATION ARE A FAMILY OF POLYNOMIALS: $f_0 = 1$, $f_1 = z$, $f_2 = \frac{1}{2}(3z^2 - 1)$, $f_3 = \frac{1}{2}(5z^3 - 3z)$ ....

LIKE SINES AND COSINES, THEY ARE ORTHOGONAL, BUT ON THE INTERVAL $[-1, 1]$

i.e. $$\int_{-1}^{1} f_m f_n \, dz = \frac{2}{2n+1} \delta_{mn}$$

ONLY THE ODD f_n SATISFY THE BOUNDARY CONDITION AT $x = 0$. YOU CAN EASILY VERIFY THAT THE OTHER B.C. IS ALWAYS SATISFIED. HENCE $\omega_1 = \Omega$, $\omega_3 = \sqrt{6}\,\Omega$ .... $\omega_n = \sqrt{\frac{n(n+1)}{2}}\,\Omega$ (n ODD)

[THE EXPERT WILL NOTE THAT THE TAYLOR EXPANSION OF $1/r'$ IN PROBLEM (8), SET 8 GAVE A SERIES OF LEGENDRE POLYNOMIALS ...]

(6) APPROXIMATE THE LOWEST FREQUENCY OF TRANSVERSE VIBRATIONS OF A BAR WHICH IS CLAMPED AT ONE END AND FREE AT THE OTHER, USING RAYLEIGH'S ENERGY METHOD.

A BRILLIANT GUESS OF RAYLEIGH IS THAT THE SHAPE $f(x)$ IS VERY NEARLY THAT WHICH IS THE SOLUTION TO THE STATICS PROBLEM WHEN YOU PUSH ON THE BAR A DISTANCE b FROM THE CLAMPED END. FOR $x > b$, THE BAR IS <u>STRAIGHT</u> (WHICH IS NEEDED TO SATISFY THE BOUNDARY CONDITION AT THE FREE END.

TO SOLVE THE STATICS PROBLEM, NOTE THAT OUR WAVE EQUATION APPLIES WHEN $\ddot{s} = 0$ — THE STATIC LIMIT!

SHOW $$f(x) = \begin{cases} 3bx^2 - x^3 & 0 < x < b \\ 3b^2x - b^3 & b < x < \ell \end{cases}$$

NEGLECT THE ROTATIONAL K.E. IN THE WAVE EQUATION,

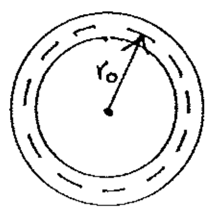
THE RESULTING EQUATION $\frac{1}{\omega^2} = g(b)$ CAN BE MAXIMIZED TO FIND THE BEST CHOICE OF $b \Rightarrow$ CUBIC EQUATION.

IT TURNS OUT THAT $b = \frac{3}{4}\ell$ IS ABOUT RIGHT. SHOW THAT THIS CHOICE LEADS TO $\omega = 3.5245\, cd/\ell^2$ (COMPARED TO $\omega = 3.5160\, cd/\ell^2$ EXACT)

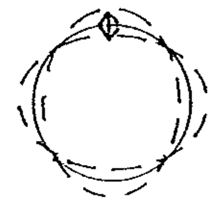
$c^2 = \frac{AY}{\rho}$ $\quad d^2 = \frac{I}{\rho A^2}$ AS IN THE NOTES.

HINT: SHOW $\langle KE \rangle = \frac{\rho A \omega^2}{4}\left(-\frac{2}{35}b^7 + b^6\ell - 3b^5\ell^2 + 3b^4\ell^3\right)$

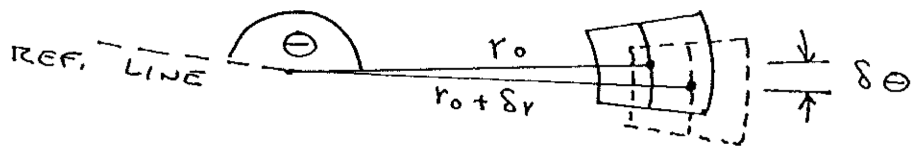
(7) AN ELASTIC RING UNDERGOES VIBRATIONS WHICH ARE IN THE PLANE OF THE RING. SUPPOSE THE VIBRATIONS DEFORM THE RING, BUT DON'T STRETCH OR COMPRESS THE LENGTH OF THE CENTER LINE, $2\pi r_0$.



THE LOWEST MODE LOOKS LIKE →



DURING THE VIBRATION, A WEDGE-SHAPED ELEMENT OF THE RING MAY MOVE BOTH RADIALLY AND AZIMUTHALLY:



THE CENTER OF THE DEFORMED ELEMENT HAS COORDINATES

$$r = r_0 + \delta r \quad , \quad \phi = \theta + \delta\theta$$

THE REQUIREMENT THAT THE CENTER LINE DOES NOT STRETCH IS THAT dS = CONSTANT WHERE $ds = r_0\, d\theta$ WHEN THERE IS NO VIBRATION. WHEN δr AND $\delta\theta \neq 0$, SHOW THIS LEADS TO

$$\frac{\delta r}{r_0} = -\frac{d}{d\theta}(\delta\theta) \qquad \text{TO FIRST ORDER}$$

BY EXPANDING ds^2 IN THE DEFORMED CASE.

CONSTRUCT THE LAGRANGIAN, IGNORING THE KINETIC ENERGY OF ROTATION. IT IS USEFUL TO NOTE THAT IN POLAR COORDINATES, THE RADIUS OF CURVATURE IS $\frac{1}{R} = \frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$ (AT LEAST APPROXIMATELY)

ELIMINATE δr IN FAVOR OF $\delta\theta$ TO SHOW

$$L = \int \left[\frac{\rho A r_0^3}{2}\left((\delta\dot\theta)^2 + (\delta\ddot\theta)^2\right) - \frac{YI}{2\rho r_0}\left(1 + \delta\theta' + \delta\theta'''\right)^2 \right] d\theta$$

ρ = VOLUME DENSITY ; A = CROSS-SECTIONAL AREA OF THE RING
I = MOMENT OF INERTIA PER UNIT LENGTH ALONG THE RING.

USE HAMILTON'S PRINCIPLE TO EXTRACT THE EQUATION OF MOTION — YOU SHOULD GET A 6TH DERIVATIVE! TRY AN OSCILLATORY SOLUTION:

$$\delta\theta = \epsilon \cos n\theta \cos\omega t \qquad (\Rightarrow \delta r = n\epsilon r_0 \sin n\theta \cos\omega t)$$

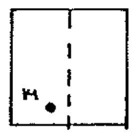
WHERE $\cos n\theta$ SATISFIES THE 'BOUNDARY CONDITION' $\delta\theta(\theta + 2\pi) = \delta\theta(\theta)$

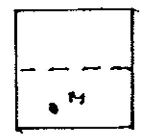
SHOW $\omega^2 = \frac{YI}{\rho^2 A r_0^4} \frac{n^2(n^2-1)^2}{n^2+1}$. THE MODES $n = 0$ & 1 ARE SURPRESSED:

$n = 0 \Rightarrow$ ROTATION OF RING, NO DEFORMATION
$n = 1 \Rightarrow$ TRANSLATION OF RING, NO DEFORMATION.

(8) CONSIDER A SQUARE DRUM HEAD OF EDGE ℓ, DENSITY ρ PER UNIT AREA, AND SURFACE TENSION T. A SMALL MASS M IS ATTACHED AT $(x, y) = (a, b)$. WHAT ARE THE 3 LOWEST FREQUENCIES?

WE CAN USE RAYLEIGH'S PERTURBATION METHOD IF WE KNOW THE PROPER FORM OF THE 'UNPERTURBED' NORMAL MODES. FOR THE (1,1) MODE THERE IS NO QUESTION. BUT THE (2,1) MODE IS DEGENERATE WITH THE (1,2) MODE. ONCE THE MASS IS ADDED, THE FORMS $f_{2,1} = \sin\frac{2\pi x}{\ell}\sin\frac{\pi y}{\ell}$ AND $f_{1,2} = \sin\frac{\pi x}{\ell}\sin\frac{2\pi y}{\ell}$

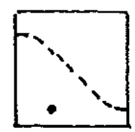




ARE NOT NORMAL MODES ANY MORE!

THE NEW MODES ARE ORTHOGONAL LINEAR COMBINATIONS OF $f_{1,2}$ AND $f_{2,1}$ SUCH THAT MASS M LIES ON A NODE OF ONE OF THE MODES:

$$f_{2a} = \frac{A f_{2,1} + B f_{1,2}}{\sqrt{A^2+B^2}} \qquad f_{2b} = \frac{B f_{2,1} - A f_{1,2}}{\sqrt{A^2+B^2}}$$



[RAYLEIGH'S METHOD MUST BE ALTERED SLIGHTLY TO DEAL WITH 2 DIMENSIONAL MODES]

SHOW THAT THE PERTURBED FREQUENCIES ARE

$$\Omega_{1,1} = \omega_{1,1}\left(1 - \frac{2M}{\rho\ell^2}\sin^2\frac{\pi a}{\ell}\sin^2\frac{\pi b}{\ell}\right)$$

$$\Omega_{2a} = \omega_{1,2} = \omega_{2,1}$$

$$\Omega_{2b} = \omega_{2,1}\left(1 - \frac{8M}{\rho\ell^2}\sin^2\frac{\pi a}{\ell}\sin^2\frac{\pi b}{\ell}\left(\cos^2\frac{\pi a}{\ell} + \cos^2\frac{\pi b}{\ell}\right)\right)$$

$\omega_{i,j} \equiv$ UNPERTURBED FREQUENCIES.

SO THE DEGENERACY HAS BEEN BROKEN BY THE PERTURBATION.

(9) CONSIDER TRANSVERSE VIBRATIONS OF A CIRCULAR MEMBRANE OF RADIUS a, SURFACE DENSITY ρ, SURFACE TENSION T.

USE $F = ma$ IN POLAR COORDINATES FOR AN AREA ELEMENT $r\,dr\,d\theta$ TO SHOW THE DISPLACEMENT $s(r, \theta, t)$ OBEYS

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} \qquad c^2 = T/\rho$$

(OR USE LAGRANGE'S METHOD)

TRY SEPARATION OF VARIABLES: $S = f(r)\,g(\theta)\,h(t)$

TO SHOW WE CAN HAVE $h = \cos\omega t$ OR $\sin\omega t$

$g = \cos n\theta$ OR $\sin n\theta$

AND $$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2}\right) f = 0$$

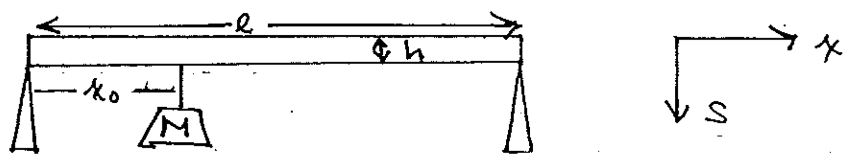
BESSEL'S EQUATION OF ORDER n. (C.F. PROB. 4)

APPLY RAYLEIGH'S METHOD TO ESTIMATE TO LOWEST NORMAL FREQUENCY ($n=0$). THE BOUNDARY CONDITIONS FOR f ARE $f(a) = 0$, $f'(0) = 0$

SHOW $\omega \approx 2.414\, c/a$ (COMPARED TO $\omega = 2.405 \frac{c}{a}$ EXACTLY)

(SEE ALSO SCIENTIFIC AMERICAN NOV. 1982)

10 A RECTANGULAR BEAM OF LENGTH ℓ, WIDTH w AND HEIGHT h IS SUPPORTED AT ITS ENDS SO THAT THE POSITIONS, BUT NOT THE SLOPES OF THE ENDS OF THE BEAM ARE FIXED. THE TWO SUPPORTS ARE AT THE SAME HEIGHT.



FROM ARGUMENTS LIKE THOSE ON P 242, WE SEE THAT THE BOUNDARY CONDITIONS ARE $S(0) = 0 = S''(0)$ ...

A MASS M IS HUNG AT A DISTANCE x_0 FROM ONE END.

GIVE A FOURIER SERIES EXPANSION FOR THE SHAPE OF THE DEFLECTION OF THE BEAM, $S = S(x)$. IGNORE THE DEFLECTION OF THE BEAM DUE TO ITS OWN WEIGHT. ALSO IGNORE ANY VARIATION IN THE DEFLECTION ACROSS THE WIDTH OF THE BEAM.

TO HELP IN EVALUATING THE FOURIER COEFICIENTS, RECALL THAT THE POTENTIAL ENERGY STORED IN THE DEFLECTED BEAM IS

$$V = \frac{YI}{2\rho}\int_0^\ell (S'')^2\,dx$$

SHOW $$S(x) = \frac{24 Mg}{\pi^4 Y}\left(\frac{\ell^3}{wh^3}\right) \underbrace{\sum_n \frac{1}{n^4} \sin\frac{n\pi x_0}{\ell} \sin\frac{n\pi x}{\ell}}$$

IF $x = x_0 = \ell/2 \longrightarrow 1 + \frac{1}{81} + \frac{1}{625} + \dots$

NOTE THAT $S_{MAX} \sim \dfrac{\ell^3}{wh^3}$

SO STIFFNESS $\sim \dfrac{1}{h^3}$

Ph 205 SET 12

7

OPTIONAL: BELOW IS A TRANSLATION OF THE BEGINNING OF A PAPER BY E. SCHRÖDINGER.

Quantisation as a Problem of Proper Values (Part I)

(*Annalen der Physik* (4), vol. 79, 1926)

§ 1. In this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the *node-numbers* of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules.

The usual form of the latter is connected with the Hamilton-Jacobi differential equation,

$$H\left(q, \frac{\partial S}{\partial q}\right) = E. \tag{1}$$

A solution of this equation is sought such as can be represented as the *sum* of functions, each being a function of one only of the independent variables q.

Here we now put for S a new unknown ψ such that it will appear as a *product* of related functions of the single co-ordinates, *i.e.* we put

$$S = K \log \psi. \tag{2}$$

The constant K must be introduced from considerations of dimensions; it has those of *action*. Hence we get

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E. \tag{1'}$$

Now we do *not* look for a solution of equation (1′), but proceed as follows. If we neglect the relativistic variation of mass, equation (1′) can always be transformed so as to become a quadratic form (of ψ and its first derivatives) equated to zero. (For the *one*-electron problem this holds even when mass-variation is not neglected.) We now seek a function ψ, such that for any arbitrary variation of it the integral of the said quadratic form, taken over the whole co-ordinate space,[1] is stationary, ψ being everywhere real, single-valued, finite, and continuously differentiable up to the second order. *The quantum conditions are replaced by this variation problem.*

First, we will take for H the Hamilton function for Keplerian motion, and show that ψ can be so chosen for *all positive*, but only for a *discrete set of negative values of E*. That is, the above variation problem has a discrete and a continuous spectrum of proper values.

The discrete spectrum corresponds to the Balmer terms and the continuous to the energies of the hyperbolic orbits. For numerical agreement K must have the value $h/2\pi$.

The choice of co-ordinates in the formation of the variational equations being arbitrary, let us take rectangular Cartesians. Then (1′) becomes in our case

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0; \tag{1''}$$

e = charge, m = mass of an electron, $r^2 = x^2 + y^2 + z^2$.

Our variation problem then reads

$$\delta J = \delta \iiint dx\,dy\,dz \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2\right] = 0, \tag{3}$$

the integral being taken over all space. From this we find in the usual way . . . .

[FOR WHAT ITS WORTH, S = ACTION = $\int L\,dt$, AND

$\frac{\partial S}{\partial q} = p$ = GENERALISED MOMENTUM. SEE L & L SEC 43.

THEN YOU CAN GO FROM SCHRÖDINGER'S (1′) TO (1″)

BY FOLLOWING HIS SUGGESTION TO REPLACE p BY $\frac{K}{\psi}\frac{\partial \psi}{\partial q}$]

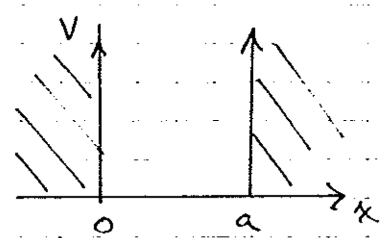
WE CONSIDER A SLIGHT VARIATION ON EQ (1″): ONE DIMENSIONAL MOTION, BUT SUBJECT TO AN ARBITRARY FORCE DERIVED FROM POTENTIAL $V(x)$. THEN

$$(1'') \rightarrow \left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{2M}{K^2}\left(E - V(x)\right)\psi^2$$

WITH THIS FORM OF THE INTEGRAND, CARRY OUT THE VARIATION SUGGESTED IN (3) TO DERIVE SCHRÖDINGER'S EQUATION.

CONSIDER THE 'INFINITE WELL' POTENTIAL

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{ELSWHERE} \end{cases}$$



SOLVE SCHRÖDINGER'S EQUATION WITH THIS POTENTIAL, SUPPOSING ψ IS CONTINUOUS AT $x=0$ AND a.

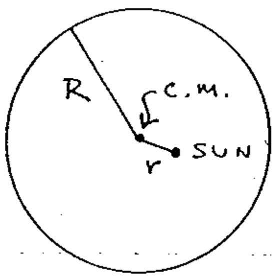
CALCULATE THE ALLOWED VALUES OF THE ENERGY E.

THIS SOLUTION IS MEANT TO DESCRIBE THE POSSIBLE MOTION OF A QUANTUM MECHANICAL PARTICLE IN A BOX. THE POSITION OF THE PARTICLE IS UNCERTAIN BY AN AMOUNT $\sim a$. THE MOMENTUM IS KNOWN — BUT NOT ITS SIGN! IN THIS SENSE, THE UNCERTAINTY

IN MOMENTUM IS $\sim p$. WHAT IS THE MINIMUM VALUE OF THE PRODUCT OF THE 'UNCERTAINTIES' pa?

(NOTE: $\psi = 0$ EVERYWHERE IS NOT CONSIDERED AN INTERESTING CASE.)

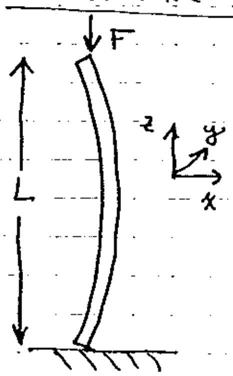
OPTIONAL II RINGWORLD



LARRY NIVEN SUGGESTED THAT A GOOD PLACE TO LIVE WOULD BE ON A LARGE CIRCULAR RING (OR BAND) WHOSE CENTER IS AT THE SUN. THE RING WOULD ROTATE ABOUT ITS C.M. TO PROVIDE AN APPARENT GRAVITY POINTING AWAY FROM THE SUN.

DISCUSS THE STABILITY OF RINGWORLD AGAINST RADIAL DISPLACEMENTS r BETWEEN THE SUN AND THE C.M. OF THE RING. CONSIDER BOTH CASES THAT THE C.M. HAS ZERO AND NON-ZERO ANGULAR MOMENTUM ABOUT THE SUN. YOU MAY ASSUME $r \ll R$ WHERE R IS THE RADIUS OF THE RING.

OPTIONAL III CHARLIE CHAPLIN'S CANE



WHEN CHARLIE LEANS ON HIS CANE IT POPS INTO A BOW SHAPE.

CONSIDER A TALL, SLENDER BEAM (THE CANE) OF LENGTH L SUBJECT TO AN APPLIED VERTICAL FORCE F. DERIVE A RELATION FOR THE MINIMUM FORCE F FOR BOWING (OR 'BUCKLING') TO OCCUR IN TERMS OF L, THE YOUNG'S MODULUS E, AND THE BENDING MOMENT OF INERTIA DEFINED BY

$$I \equiv \int x^2 \, dx \, dy$$

YOU MAY IGNORE GRAVITY AND THE COMPRESSION OF THE BEAM'S LENGTH L. THE ENDS OF THE BEAM ARE FREE TO ROTATE.

HINT: DO NOT USE ENERGY METHODS; RATHER, CONSIDER THE TORQUE EQUATION FOR STATIC EQUILIBRIUM OF THE UPPER PART OF THE BUCKLED BEAM.

SHOW THAT $F = EI\left(\frac{\pi}{L}\right)^2$ IS THE CRITICAL FORCE (EULER)