PH 205 SET 10

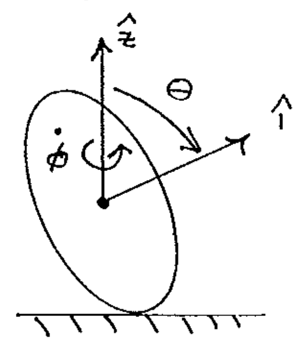
DUE: THURSDAY DEC 8, 1988 ; MAXIMUM RECORDED SCORE = 50 POINTS

① SPINNING COIN REVISITED

IF YOU SPIN A COIN ON A TABLE, YOU CAN MAKE IT SPIN ABOUT A VERTICAL DIAMETER AS THE AXIS, WITH THE C.M. AT REST. IF THE ANGULAR VELOCITY BECOMES 'TOO LOW', THE COIN FALLS OVER INTO THE MOTION OF PROB. ⑥, SET 9.

CONSIDER A DISC OF MASS M, RADIUS a WHICH SPINS WITHOUT FRICTION ON A HORIZONTAL SURFACE ($\Rightarrow$ C.M. CAN ONLY MOVE VERTICALLY). USE EULER'S ANGLES AND LAGRANGE'S METHOD TO FIND THE STEADY PRECESSION RATE $\dot{\phi}$ AS A FUNCTION OF THE ANGLE θ OF THE SYMMETRY AXIS TO THE VERTICAL, AND OF ω_1 = ANGULAR VELOCITY ABOUT THE SYMMETRY AXIS

SHOW $$\dot{\phi}_{STEADY} = \frac{\omega_1 \pm \sqrt{\omega_1^2 + \dfrac{4g\cos^2\theta}{a\sin\theta}}}{\cos\theta}$$



AS $\theta \to \pi/2$ (COIN SPINNING ON EDGE) THE ABOVE SOLUTION BREAKS DOWN! SHOW THAT WHEN $\theta = \pi/2$ THE TWO POSSIBLE MOTIONS ARE

$$\begin{cases} \omega_1 \text{ ARBITRARY}, \ \dot{\phi} = 0 \iff \text{'ROLLING \& SLIPPING'} \\ \omega_1 = 0, \ \dot{\phi} \text{ ARBITRARY} \iff \text{'SPINNING ON EDGE'} \end{cases}$$

WE ARE INTERESTED IN THE 2ND CASE.

USE THE SMALL ANGLE APPROXIMATION ($\theta = \pi/2 - \epsilon$)

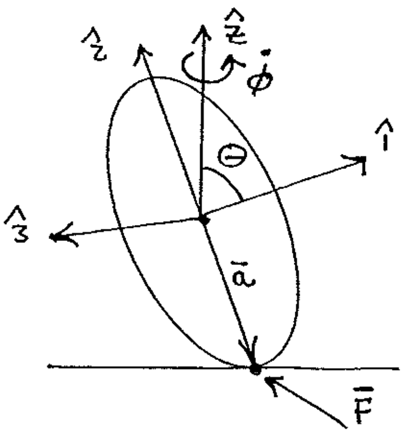
TO SHOW THE 2ND MOTION IS <u>STABLE</u> ONLY IF

$$\dot{\phi} > 2\sqrt{\frac{g}{a}}$$

OTHERWISE THE COIN FALLS OVER....

(2) Rolling Disc Revisited

We consider arbitrary motion of a disc rolling without slipping on a horizontal plane. Experts will note that this is a non-holonomic problem with 5 coordinates and 2 constraints. But try it without resort to Lagrange multipliers!



The disc has mass M, radius a

Choose a coordinate system similar to, but not quite the same, as Euler's.

$\hat{1}$ along the symmetry axis

$\hat{z}$ vertical

$\hat{2}$ in the plane of the disc, and in $\hat{1}$-$\hat{z}$ plane

$\hat{3} = \hat{1} \times \hat{2}$ is horizontal, & in plane of disc.

This is a principal axis coord. system $\Rightarrow$ Inertia Tensor Diagonal! But it is not quite the 'body frame'!

θ = angle between $\hat{1}$ & $\hat{z}$

$\dot{\phi}$ = rotation rate of the disc about the vertical.

$\bar{F}$ = unknown force on the disc at the point of contact.

The elementary equations of motion are, of course,

$$\bar{F}_{TOTAL} = m \frac{d\bar{v}}{dt} \quad \text{where } \bar{v} = \bar{v}_{\text{of cm}} \quad ; \quad \bar{N}_{c.m.} = \frac{d\bar{L}_{cm}}{dt}$$

The constraint of rolling without slipping is easily written down in terms of velocities:

$$\bar{v}_{\text{of contact}} = 0 = \bar{v} + \bar{\omega} \times \bar{a} \qquad \text{(Chasles' Theorem)}$$

where $\bar{a}$ = vector from c.m. to point of contact

$\bar{\omega}$ = total angular velocity $= \bar{\omega}_a + \dot{\psi}\,\hat{1}$

$\bar{\omega}_a$ = angular velocity of the axes

$\dot{\psi}$ = angular velocity of the disc relative to the $\hat{1},\hat{2},\hat{3}$ axes

Define $\omega_1 \equiv \dot{\psi} + \omega_{a_1}$ to avoid carry both symbols thru the entire problem.

ELIMINATE $\bar{F}$ AND $\bar{v}$ TO GET THE EQUATION OF MOTION

$$\frac{ma^2}{4}\frac{d}{dt}\left(2\omega_1\hat{1} + \dot{\phi}\sin\theta\,\hat{2} - \dot{\theta}\,\hat{3}\right) =$$
$$mga\cos\theta\,\hat{3} - ma^2\,\hat{2}\times\frac{d}{dt}\left(\dot{\theta}\,\hat{1} + \omega_1\hat{3}\right)$$

FIRST CONSIDER <u>STEADY</u> MOTION $\Rightarrow$ $\omega_1, \dot{\phi}$ = CONSTANT
$\dot{\theta} = 0$

NOTE THAT FOR THE AXES, $\frac{d\hat{i}}{dt} = \bar{\omega}_a \times \hat{i}$ TO SHOW THAT

$$\dot{\phi}^2\sin\theta\cos\theta - 6\omega_1\dot{\phi}\sin\theta - \frac{4g}{a}\cos\theta = 0$$

[TO RELATE THIS TO PROB ⑤, SET 9, NOTE THAT $a\omega_1 = -b\dot{\phi}$ FOR ROLLING IN A CIRCLE OF RADIUS b, AND THAT θ IS DEFINED DIFFERENTLY.]

AT $\theta = \pi/2$ THE DISC IS ON EDGE.

WE ARE INTERESTED IN THE <u>ROLLING</u> SOLUTION: $\dot{\phi} = 0$, BUT ω_1 ARBITRARY.

IS THIS MOTION STABLE?

THAT IS, CONSIDER $\theta = \pi/2 - \epsilon$, $\dot{\phi}$ SMALL AND ω_1 ARBITRARY.

PLUG INTO THE EQUATION OF MOTION AND IGNORE 2ND ORDER TERM: ϵ^2, $\dot{\epsilon}\dot{\phi}$ ETC.

REMEMBER $\frac{d\hat{i}}{dt} = \bar{\omega}_a \times \hat{i}$

SHOW THAT THE $\hat{1}$ TERMS $\Rightarrow$ $\dot{\omega}_1 = 0$, ω_1 = CONST

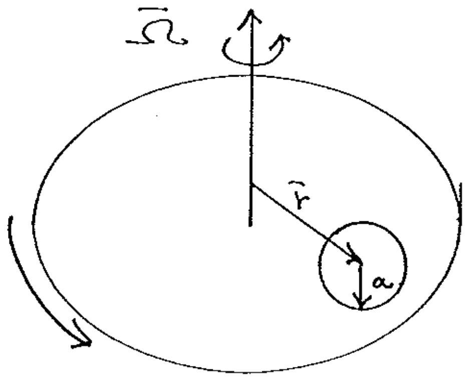
$\hat{2}$ TERMS $\Rightarrow$ $\dot{\phi} = -2\omega_1\epsilon$

$\hat{3}$ TERMS $\Rightarrow$ $\ddot{\epsilon} = -\frac{4}{5}\left(3\omega_1^2 - \frac{g}{a}\right)\epsilon$

SO THE ROLLING IS STABLE IF $\omega_1 > \sqrt{g/3a}$

IF $\omega_1 < \sqrt{g/3a}$ THE DISC FALLS OVER AND GOES INTO THE MOTION OF PROB ⑥, SET 9. TRY IT WITH A COIN, A HULA HOOP....

③ <u>MARBLE</u> <u>ROLLING ON A TURNTABLE</u> (SO THAT'S HOW YOU LOST YOUR MARBLES)



A MARBLE OF MASS m, RADIUS a (A UNIFORM SPHERE) ROLLS WITHOUT SLIPPING ON A HORIZONTAL TURNTABLE. THE TURNTABLE ROTATES WITH CONSTANT ANGULAR VELOCITY $\bar{\Omega}$ ABOUT THE VERTICAL.

USE THE APPROACH OF PROB ② TO ANALYZE THE MOTION BY 'ELEMENTARY' METHODS IN THE <u>LAB</u> FRAME.

a) WHAT IS THE CONSTRAINT RELATION BETWEEN $\bar{\Omega}$, $\bar{r}$, $\bar{v}$, $\bar{a}$ AND $\bar{\omega}$?

$\bar{r}$ = C.M. POSITION OF MARBLE

$\bar{v}$ = C.M. VELOCITY OF MARBLE

$\bar{a}$ = RADIUS VECTOR FROM THE C.M. TO POINT OF CONTACT

$\bar{\omega}$ = TOTAL ANGULAR VELOCITY.

b) IT SEEMS REASONABLE THAT A POSSIBLE MOTION IS JUST MOTION IN A CIRCLE OF RADIUS r ABOUT THE TURNTABLE AXIS. USE 'ELEMENTARY METHODS' TO SHOW THAT

$$\omega = \frac{\Omega}{a}\left(\frac{ma^2}{I+ma^2}\right) r$$

[WE NEGLECT A POSSIBLE SPIN OF THE MARBLE ABOUT THE VERTICAL, WHICH IS INDEPENDENT OF Ω...]

WHERE I = MOMENT OF INERTIA OF THE MARBLE.

IN WHAT DIRECTION IS $\bar{\omega}$? YOU SHOULD ALSO FIND $\dot{\phi} = \Omega \frac{I}{I+ma^2}$ FOR C.M. MOTION ABOUT THE VERTICAL

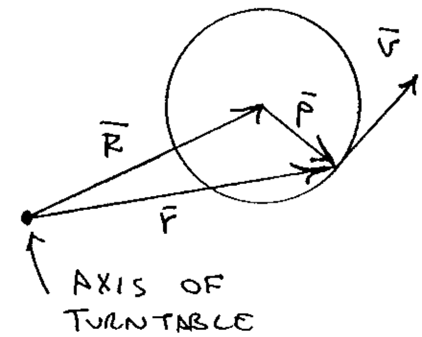
c) CONSIDER ARBITRARY MOTION. NOTE THAT BY DIFFERENTIATING THE CONSTRAINT RELATION, YOU CAN ELIMINATE $d\bar{\omega}/dt$ FROM THE EQUATIONS OF MOTION LEADING TO

$$\frac{I+ma^2}{I} \cdot \frac{d\bar{v}}{dt} = \frac{d}{dt}(\bar{\Omega} \times \bar{r})$$

WITH $\bar{\Omega}$ = CONST, SHOW THAT THIS LEADS TO

$$\bar{v} = \frac{I}{I+ma^2}\, \bar{\Omega} \times \bar{\rho} \qquad \text{WHERE} \qquad \bar{r} = \bar{R} + \bar{\rho}$$

AND $$\bar{R} = \bar{r}_0 + \frac{I+ma^2}{I}\, \frac{\bar{\Omega} \times \bar{v}_0}{\Omega^2}$$

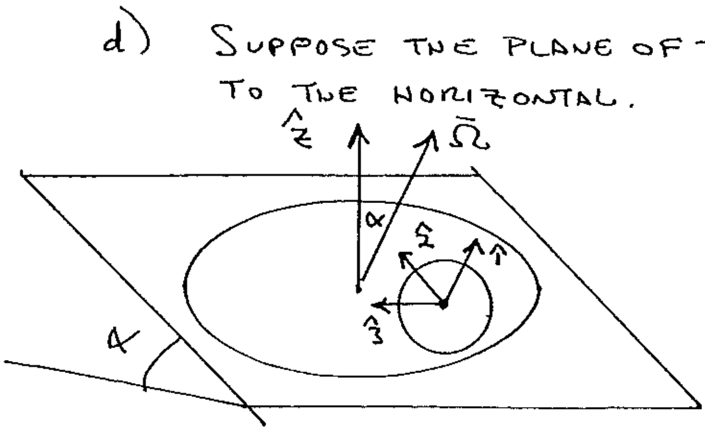


THIS IS <u>CIRCULAR</u> MOTION IN A CIRCLE OF RADIUS ρ ABOUT THE POINT AT $\bar{R}$.

$\bar{v}_0$, $\bar{r}_0$ = INITIAL VELOCITY AND POSITION OF THE MARBLE.

FINALLY, SHOW
$$\bar{\omega} = \frac{\Omega}{a}\,\frac{I\bar{R} + Ma^2\bar{r}}{I + Ma^2}$$

d) SUPPOSE THE PLANE OF THE TURN-TABLE MAKES ANGLE α TO THE HORIZONTAL.



TAKE INTO ACCOUNT THE POSSIBILITY THAT THE FORCE $\bar{F}$ OF THE TURN TABLE ON THE MARBLE INCLUDES A NORMAL FORCE $\neq Mg$.

CHOOSE AXES $\hat{1}$ $\perp$ TO TURNTABLE

$\hat{2}$ POINTING UP SLOPE

$\hat{3}$ HORIZONTAL.

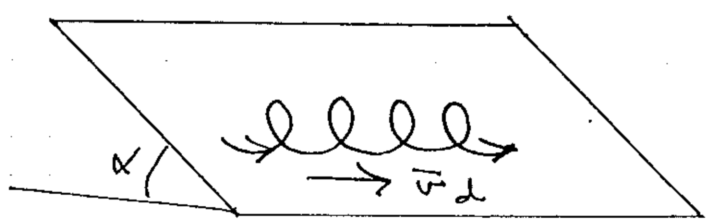
FIND THE SLIGHTLY MODIFIED EQUATION OF MOTION FOR $\frac{d\bar{v}}{dt}$ AS IN PART c)

SHOW THAT A SOLUTION IS $\bar{v} = \bar{v}_c + \bar{v}_d$

WHERE $\bar{v}_c$ = THAT FOUND IN PART c)

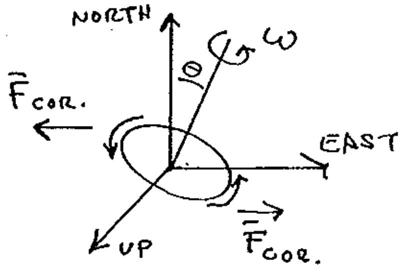
$$\bar{v}_d = -\frac{Mg\,a^2 \sin\alpha}{I\Omega}\,\hat{3} = \text{CONSTANT}$$

HENCE THE MOTION LOOKS LIKE



WHY NOT TRY IT!

④ GYROCOMPASS



TOP VIEW

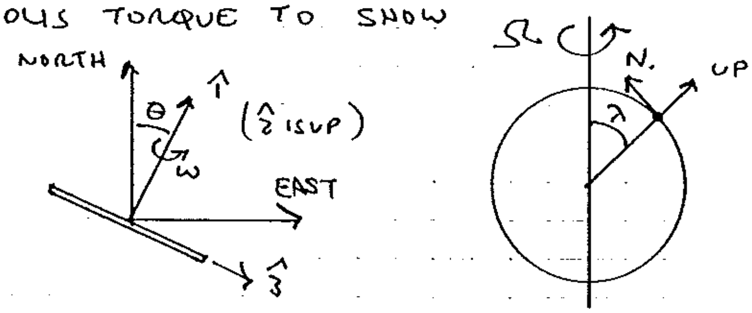
A GYROCOMPASS IS A SPINNING FLYWHEEL WHOSE AXIS OF ROTATION IS CONSTRAINED TO MOVE IN A HORIZONTAL PLANE AT THE SURFACE OF THE EARTH. (NO HORIZONTAL FORCES ARE APPLIED TO THE AXIS).

IF WE ANALYZE THE MOTION IN A FRAME FIXED ON THE EARTH'S SURFACE, WE MUST TAKE THE CORIOLIS FORCE INTO ACCOUNT. WHEN THE AXIS MAKES ANGLE θ TO THE NORTH AS SHOWN, THE LEFT SIDE OF THE FLYWHEEL IS MOVING UP ⇒ CORIOLIS FORCE TO THE WEST. THE RIGHT SIDE OF THE FLYWHEEL IS MOVING DOWN ⇒ FORCE TO THE EAST.

∴ THERE IS A NET TORQUE ABOUT THE VERTICAL WHICH TENDS TO RESTORE θ TO 0°. I.E., THE GYRO LIKES TO POINT NORTH!

a) ASSUME THE FLYWHEEL IS A HOOP OF RADIUS a. CALCULATE THE TOTAL CORIOLIS TORQUE TO SHOW

$$N_1 = 0$$
$$N_2 = M a^2 \omega \Omega \sin\lambda \sin\theta$$
$$N_3 = M a^2 \omega \Omega \cos\lambda$$



WHERE $\hat 1, \hat 2, \hat 3$ ARE PRINCIPAL AXES AS SHOWN. (THESE ARE NOT BODY AXES, IN THAT $\hat 2$ IS ALWAYS UP)
Ω = ROTATION OF EARTH
λ = CO-LATITUDE OF THE GYROCOMPASS.

SHOW THAT THIS TORQUE CAUSES THE GYRO TO OSCILLATE ABOUT $\theta = 0°$ WITH ANGULAR FREQUENCY $\sqrt{2\omega\Omega\sin\lambda}$ (FOR SMALL θ_{MAX}).

b) ANALYZE THE GYROCOMPASS FROM THE POINT OF VIEW OF AN INERTIAL OBSERVER.

SUCH AN OBSERVER WOULD SAY $\frac{d\bar L}{dt} = \bar N$, WHERE TORQUE $\bar N$ IS ONLY DUE TO THE CONSTRAINT FORCES ON THE AXLE, WHICH KEEP IT HORIZONTAL. THESE DON'T MAKE THE GYRO POINT NORTH.

BUT $\bar L = \bar{\bar I} \cdot \bar\omega_{TOT}$ AND $\bar\omega_{TOT}$ HAS 3 PIECES.

- ROTATION ABOUT THE GYRO AXLE BY AMOUNT $\bar\omega$
- ROTATION $\dot\theta$ IN THE PLANE OF THE SURFACE OF THE EARTH.
- ROTATION OF THE EARTH ABOUT ITS AXIS, $\bar\Omega$

EVALUATE $\bar{\omega}_{TOT}$ IN THE INERTIAL FRAME BY GIVING ITS COMPONENTS ALONG THE DIRECTIONS OF THE PRINCIPAL AXES DEFINED ABOVE. ALSO EVALUATE I FOR A FLYWHEEL WHICH IS A UNIFORM DISC OF MASS M, RADIUS a.

WITH RESPECT TO THE INERTIAL OBSERVER, THE PRINCIPAL AXES ARE ROTATING WITH ANGULAR VELOCITY $\dot{\bar{\Theta}} + \bar{\Omega}$.

EVALUATE $\bar{N} = \dfrac{d\bar{L}}{dt}$.

SHOW THAT THE COMPONENTS OF THIS EQUATION FOR WHICH $N_i = 0$ LEAD TO $\dot{\omega} = 0$ AND $\ddot{\Theta} + 2\omega\Omega(\sin\lambda)\Theta = 0$ FOR Θ & Ω SMALL.

HENCE THE GYRO OSCILLATES ABOUT $\Theta = 0$ (NORTH) WITH ANGULAR FREQUENCY $\sqrt{2\omega\Omega\sin\lambda}$ AS IN PART a).

IN PRACTICE A MOTOR IS REQUIRED TO KEEP ω CONSTANT. DAMPING IN THE GYRO KILLS THE OSCILLATIONS, LEAVING THE AXIS POINTING NORTH.

OF COURSE, IF THE CONSTRAINING PLANE OF THE GYRO TIPS, THE GYRO WON'T POINT NORTH ANYMORE. A REAL GYROCOMPASS MUST HAVE A MECHANISM TO FIND THE HORIZONTAL PLANE EVEN WHEN ITS SUPPORTS ARE TIPPING — AS ON A SHIP OR AIRPLANE. THIS REQUIRES A PLUMB BOB — AND A MECHANISM TO DEFEAT THE INFLUENCE OF A POSSIBLY OSCILLATING POINT OF SUPPORT....

WELL, IT'S A BIT COMPLICATED.

(5) CONSIDER AN ASYMMETRIC TOP WHOSE PRINCIPAL MOMENTS OF INERTIA ARE $I_1 > I_2 > I_3$. WE HAVE CLAIMED THAT ROTATIONS WITH $\bar{\omega}$ NEAR $\hat{2}$ ARE 'UNSTABLE'.

EXAMINE THE SPECIAL CASE WHEN $T = \dfrac{L^2}{2I_2}$ (= KINETIC ENERGY)

USE THE EXPRESSIONS FOR T AND L^2 TO SHOW

$$\omega_1^2 = \frac{I_2 - I_3}{I_1 - I_3}\,\frac{L^2 - I_2^2\omega_2^2}{I_1 I_2} \qquad \omega_3^2 = \frac{I_1 - I_2}{I_1 - I_3}\,\frac{L^2 - I_2^2\omega_2^2}{I_2 I_3}$$

AND THAT EULER'S EQUATIONS LEAD TO

$$\omega_2 = \omega_{2\,MAX} \tanh\left[K\,\omega_{2\,MAX}\,(t+t_0)\right] \quad \text{WHERE } \omega_{2\,MAX} = L/I_2$$

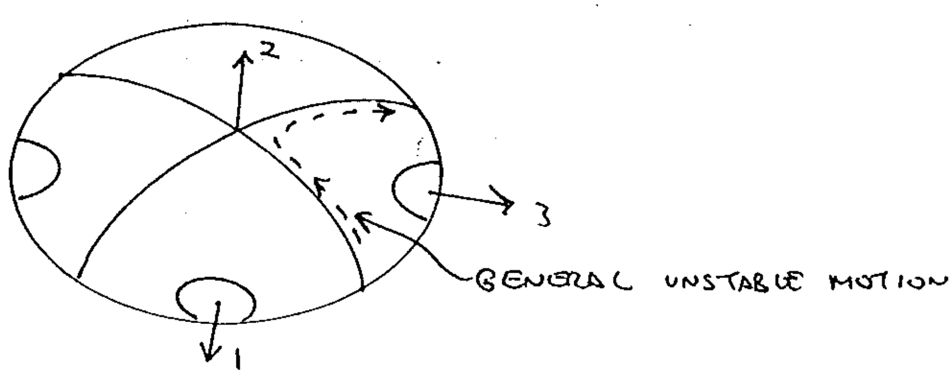
AND $K = \sqrt{\dfrac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}}$

AND $\omega_1 = \omega_{1\,MAX} \sqrt{\dfrac{I_1(I_2 - I_3)}{I_2(I_1 - I_3)}}\ \mathrm{sech}\left[K\,\omega_{2\,MAX}\,(t+t_0)\right]$

$$\omega_3 = \omega_{3\,MAX} \sqrt{\frac{I_3(I_1 - I_2)}{I_2(I_1 - I_3)}}\ \mathrm{sech}\left[K\,\omega_{2\,MAX}\,(t+t_0)\right]$$

HENCE AS $t \to \infty$, $\omega_1, \omega_3 \to 0$, $\omega_2 \to \omega_{2\,MAX}$ AND THE ROTATION IS ABOUT AXIS 2.

THUS FOR THIS SPECIAL CASE OF MOTION ALONG THE 'SEPARATING POLHODES', A KIND OF STABILITY OCCURS.



IN PRACTICE THIS IS HARD TO ACHIEVE, SINCE IF $T \neq \dfrac{L^2}{2I_2}$

$\bar{\omega}$ WILL MOVE TOWARDS $\hat{2}$ & THEN 'BOUNCE' AWAY, MOVING TOWARDS $-\hat{2}$, ROUGHLY ALONG THE OTHER SEPARATING POLHODE ...