Ph 205 LECTURE 8

OSCILLATIONS ABOUT THE EQUILIBRIUM POINT OF ONE COORDINATE

MANY INTERESTING PROBLEMS OF MOTION IN ONE OR MORE DIMENSIONS CAN BE REDUCED TO A CASE OF SPRING-LIKE OSCILLATIONS ABOUT THE EQUILIBRIUM POINT OF SOME ONE COORDINATE.

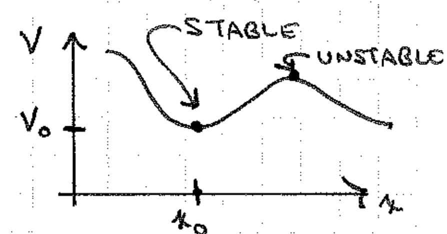
IF SO, CERTAIN TRICKS SUITABLE TO ONE DIMENSIONAL MOTION CAN BE APPLIED. THESE ARE SKETCHED IN B & O SEC 2-3, AND L & L SEC 11.

IN BRIEF, MOTION IN ONE DIMENSION DUE TO A CONSERVATIVE FORCE $F(x) = -\frac{dV(x)}{dx}$ OBEYS ENERGY CONSERVATION: $E = \frac{1}{2}mv^2 + V(x) = \text{CONST.}$

THIS ALLOWS A FORMAL SOLUTION $t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}} = t(x)$

WHICH CAN BE INVERTED TO FIND $x(t)$ IN PRINCIPLE.

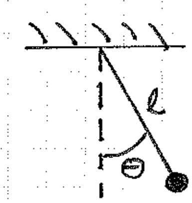
IN MANY CASES THE POTENTIAL $V(x)$ HAS A LOCAL MINIMUM - AN EQUILIBRIUM POINT. IN THE VICINITY OF THE MINIMUM x_0, $V(x) = V_0 + \frac{1}{2}(x - x_0)^2 V''(x_0) + \ldots$



THE PARABOLIC PART OF THIS TAYLOR EXPANSION OF THE POTENTIAL CORRESPONDS TO A SPRING-LIKE FORCE $F = -k\,\Delta x$ WHERE $k = V''(x_0)$

WE WILL EXPLORE THIS APPROXIMATION QUITE EXTENSIVELY. THE BASIC EXAMPLE IS THE SIMPLE PENDULUM, DISCUSSED IN B & O SEC.S 2-8, 9 AND L & L p. 26.

A PROMINENT QUESTION IS HOW DOES THE PERIOD OF OSCILLATION DEPEND ON THE AMPLITUDE? RECALL THAT FOR SIMPLE HARMONIC MOTION ($\Leftrightarrow$ PURE PARABOLIC POTENTIAL) THE PERIOD IS INDEPENDENT OF THE AMPLITUDE.



NOW $V(\theta) = mg\ell(1 - \cos\theta)$, AND AT θ_{MAX}, K.E. = 0

SO $E = mg\ell(1 - \cos\theta_{MAX})$

THE PERIOD IS $T = 4\sqrt{\frac{m}{2}} \int_0^{\theta_{MAX}} \frac{\ell\, d\theta}{\sqrt{E - V(\theta)}} = 2\sqrt{\frac{2\ell}{g}} \int_0^{\theta_{MAX}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{MAX}}}$

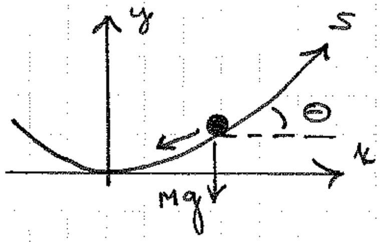
AFTER THE DEVIOUS SUBSTITUTION $\sin\beta = \frac{\sin\theta/2}{\sin\frac{\theta_{MAX}}{2}}$

WE FIND $T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_{MAX}^2}{16} + \ldots\right) > 2\pi\sqrt{\ell/g}$

WE WILL DERIVE THIS RESULT BY OTHER METHODS LATER IN THE COURSE.

THE TAUTOCHRONE

WHAT KIND OF PENDULUM WOULD HAVE ITS PERIOD TRULY INDEPENDENT OF ITS AMPLITUDE? IN 1671, C. HUYGENS REALIZED THAT THIS QUESTION IS EQUIVALENT TO: FOR WHAT SMOOTH CURVE IS THE TIME OF FALL OF A FRICTIONLESS SLIDING MASS INDEPENDENT OF INITIAL POSITION?



WE MEASURE DISTANCE S ALONG OUR DESIRED CURVE. THEN NEWTON'S LAWS TELL US THE MOTION OBEYS

$$m\ddot{s} = -mg\sin\theta$$

TO HAVE THE PERIOD INDEPENDENT OF THE AMPLITUDE, WE NEED SIMPLE HARMONIC MOTION

$$m\ddot{s} = -ks$$

HENCE THE CURVE MUST OBEY $s = C\sin\theta$, C = CONSTANT

WE CAN CONVERT THIS INTO A DESCRIPTION OF THE CURVE IN TERMS OF x AND y:

$$\frac{dx}{d\theta} = \frac{dx}{ds}\frac{ds}{d\theta} = \cos\theta \cdot C\cos\theta$$

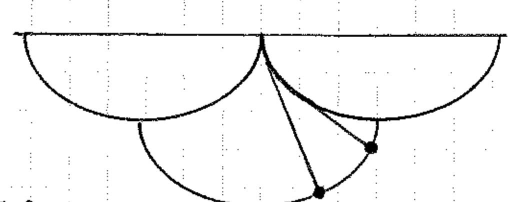
$$x = \int C\cos^2\theta \, d\theta = \frac{C}{2}\int(1+\cos 2\theta)\,d\theta = \frac{C}{4}(2\theta + \sin 2\theta)$$

$$\frac{dy}{d\theta} = \frac{dy}{ds}\frac{ds}{d\theta} = \sin\theta \cdot C\cos\theta$$

$$y = \int C\sin\theta\cos\theta\,d\theta = \frac{C\sin^2\theta}{2} = \frac{C}{4}(1-\cos 2\theta)$$

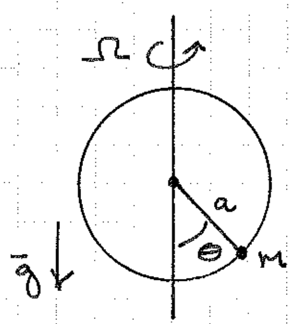
WHICH ARE THE EQUATIONS OF A CYCLOID!

HUYGENS CONSTRUCTED ACTUAL CLOCKS USING THIS DEDUCTION, PLUS THE INTERESTING RESULT THAT A PENDULUM BOB CAN BE MADE TO MOVE IN A CYCLOIDAL PATH IF THE STRING WRAPS AROUND ANOTHER CYCLOID.



MOST PRACTICAL PENDULUM CLOCKS USE A WEIGHT-DRIVEN 'ESCAPEMENT' MECHANISM TO KEEP THE AMPLITUDE CONSTANT. THEN THE SIMPLE PENDULUM CONSTRUCTION SUFFICES.

EXAMPLE



A BEAD OF MASS M SLIDES ON A CIRCULAR HOOP OF RADIUS a. THE HOOP IS CONSTRAINED TO ROTATE WITH CONSTANT ANGULAR VELOCITY Ω ABOUT A VERTICAL DIAMETER. (THIS PROBLEM IS SIMILAR TO 10 d) OF SET 1, WITH GRAVITY ADDED.)

"CENTRIFUGAL FORCE" PUSHES THE BEAD OUTWARDS, WHILE GRAVITY PULLS IT DOWN. THUS THERE MAY BE AN EQUILIBRIUM ANGLE $\theta_0 > 0$. IF SO, THEN WE CAN HAVE SMALL OSCILLATIONS IN θ ABOUT θ_0.

OF COURSE, THE WHOLE SYSTEM IS ROTATING, SO THE MOTION OF THE BEAD THRU SPACE IS COMPLICATED, AND MAY NOT BE PERIODIC IN SPACE.

BUT THERE IS ONLY 1 DEGREE OF FREEDOM, θ. SO WE CAN USE OUR 1-DIMENSIONAL TECHNIQUES.

$$L = T - V = \tfrac{1}{2} m \left(a^2 \dot{\theta}^2 + a^2 \sin^2\theta\, \Omega^2 \right) - mga(1-\cos\theta)$$

IS $E = T + V$ CONSERVED? $\frac{\partial L}{\partial t} = 0 \Rightarrow H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \text{CONSTANT}$

$$H = \tfrac{1}{2} m \left(a^2 \dot{\theta}^2 - a^2 \sin^2\theta\, \Omega^2 \right) + mga(1-\cos\theta) \neq E \quad !$$

WE HAVE A MOVING CONSTRAINT, SO THE CONSTRAINT FORCE DOES WORK AND E IS NOT CONSERVED. WE FACE THE MUSIC AND DIFFERENTIATE TO GET LAGRANGE'S EQUATION OF MOTION:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ma^2\ddot{\theta} = \frac{\partial L}{\partial \theta} = ma^2\Omega^2 \sin\theta\cos\theta - mga\sin\theta$$

$\ddot{\theta} = 0$ IS THE CONDITION FOR EQUILIBRIUM.

SO WE HAVE $\theta_0 = 0$ AND $\cos\theta_0 = \frac{g}{a\Omega^2}$ AS EQUILIBRIUM POINTS.

ARE THESE STABLE OR NOT?

ONE WAY TO ANSWER THIS IS VIA AN EXTREMELY USEFUL TRICK.

THE EQUATION $ma^2\ddot{\theta} = \frac{\partial L}{\partial \theta}$ IS LIKE $M_{EFF}\ddot{x} = -\frac{\partial V_{EFF}}{\partial x} = F_{EFF}$

(EFF $\equiv$ EFFECTIVE)

OUR PROBLEM IS FORMALLY LIKE THAT OF A POINT MASS $M_{EFF} = ma^2$ IN A ONE-DIMENSIONAL FORCE FIELD

$$F_{EFF} = ma^2\Omega^2 \sin\theta\cos\theta - mga\sin\theta$$

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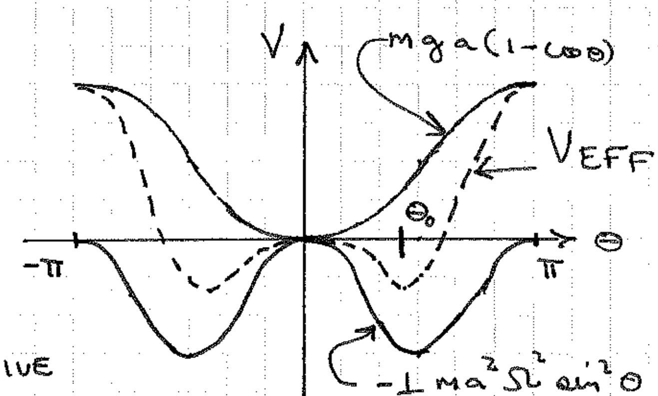
THIS F_{EFF} IS DERIVABLE FROM AN EFFECTIVE POTENTIAL

$$V_{EFF} = -\frac{1}{2} m a^2 \Omega^2 \sin^2\theta + mga(1-\cos\theta)$$
$$= V_{CENTRIFUGAL} + V_{TRUE}$$

IN PROBLEMS WITH A CONSTANT ANGULAR VELOCITY, THE CENTRIFUGAL POTENTIAL CAN BE REMEMBERED AS $V_{CENT} = -\frac{1}{2} m\Omega^2 r^2$, CORRESPONDING TO $F_{CENT} = m\Omega^2 r$.

IF WE SKETCH V_{EFF} WE CAN QUICKLY DECIDE ABOUT THE STABILITY OF THE EQUILIBRIUM.

CASE A. $\cos\theta_0 = \frac{g}{a\Omega^2} < 1$

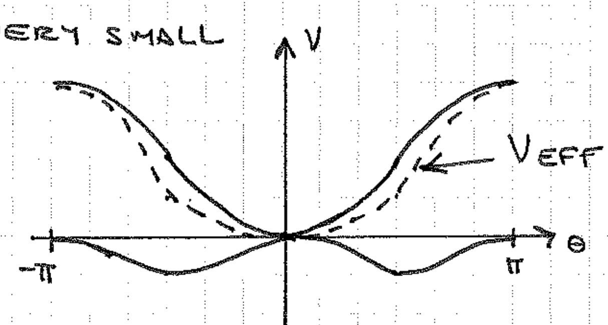


CLEARLY $\theta_0 = 0$ IS UNSTABLE

AND $\cos\theta_0 = \frac{g}{a\Omega^2}$ IS STABLE

(ANALYTICALLY, THE 2ND DERIVATIVE OF THE EFFECTIVE POTENTIAL MUST BE POSITIVE AT θ_0 FOR STABILITY.)

CASE B: $\frac{g}{a\Omega^2} > 1 \iff \Omega$ VERY SMALL



V_{EFF} NEVER GOES NEGATIVE;

$\theta_0 = 0$ IS THE ONLY EQUILIBRIUM POINT, AND IT IS STABLE

WE CAN ALSO DEFINE A KIND OF EFFECTIVE KINETIC ENERGY:

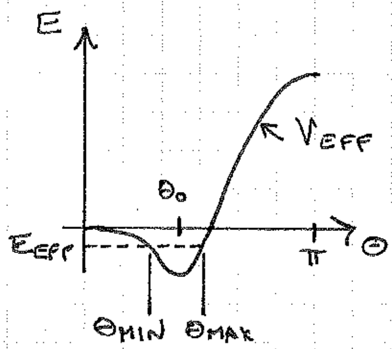
$$T_{EFF} = \frac{1}{2} M_{EFF}\, \dot\theta^2 = \frac{1}{2} m a^2 \dot\theta^2 = \text{K.E. OF THE ONE}$$

DIMENSIONAL MOTION IN θ ONLY.

WE NOTE THAT $E_{EFF} = T_{EFF} + V_{EFF} = H = \text{CONSTANT}$!

THIS OBSERVATION SUGGESTS THAT WE COULD HAVE SPLIT THE HAMILTONIAN INTO T_{EFF} & V_{EFF} WITHOUT EVER HAVING DERIVED THE EQUATIONS OF MOTION.

BY CONSIDERING THE VARIOUS EFFECTIVE ENERGIES WE CAN DISCUSS THE CHARACTER OF THE MOTION — WITHOUT AN EXACT INTEGRATION OF THE EQUATIONS OF MOTION.



FOR A GIVEN CONSTANT $E_{EFF} = T_{EFF} + V_{EFF}$, THE BEAD CAN SLIDE ONLY BETWEEN THE ANGLES θ_{MIN} AND θ_{MAX} WHERE $T_{EFF} = 0$.

IF $E_{EFF} > 0$ THE MOTION WILL CROSS $\theta = 0$ ($\theta_{MIN} = -\theta_{MAX}$), BUT THE VELOCITY $a\dot{\theta}$ WILL BE GREATER AT θ_0 THAN AT $\theta = 0$.

WE CONCLUDE OUR ANALYSIS BY CONSIDERING SMALL OSCILLATIONS ABOUT EQUILIBRIUM.

<u>CASE A</u>. $\cos\theta_0 = \dfrac{g}{a\Omega^2} < 1$

IF WE EXPAND THE POTENTIAL ABOUT θ_0,

$$V_{EFF}(\theta) = V_{EFF}(\theta_0) + \frac{1}{2}\frac{d^2V_{EFF}(\theta_0)}{d\theta^2}(\theta-\theta_0)^2 + \cdots$$

THE 2ND DERIVATIVE AT θ_0 IS LIKE A SPRING CONSTANT

$$K_{EFF} = \frac{d^2V_{EFF}(\theta_0)}{d\theta^2}$$

THE EQUATION OF MOTION IS $M_{EFF}\ddot{\theta} = Ma^2\ddot{\theta} = -K_{EFF}(\theta-\theta_0) - \cdots$

FOR $\theta - \theta_0$ SMALL WE CAN SOLVE THIS AS

$$\theta = \theta_0 + A\cos\omega t \quad \text{WITH} \quad \omega = \sqrt{\frac{K_{EFF}}{M_{EFF}}}$$

FOR THE PRESENT CASE, $\dfrac{dV_{EFF}}{d\theta} = -ma^2\Omega^2\underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta} + mga\sin\theta$

$$\frac{d^2V_{EFF}}{d\theta^2} = -ma^2\Omega^2(2\cos^2\theta - 1) + mga\cos\theta$$

AT θ_0, $K_{EFF} = -ma^2\Omega^2\left(\dfrac{2g^2}{a^2\Omega^4} - 1\right) + mga\dfrac{g}{a\Omega^2} = Ma^2\left(\Omega^2 - \dfrac{g^2}{a^2\Omega^2}\right)$

AND $\omega = \sqrt{\dfrac{K_{EFF}}{M_{EFF}}} = \Omega\sqrt{1 - \left(\dfrac{g}{a\Omega^2}\right)^2} = \Omega\sin\theta_0 < \Omega$

THE PERIOD OF <u>OSCILLATION</u> IS LONGER THAN THE PERIOD OF ROTATION. THE MOTION IS PERIODIC IN <u>SPACE</u> ONLY IF $\dfrac{\Omega}{\omega} = 1, 2, 3 \ldots$

I.E. IF $\sin\theta_0 = 1, \frac{1}{2}, \frac{1}{3} \ldots$

CASE B. $\frac{g}{a\Omega^2} > 1 \Rightarrow \Theta_0 = 0$ AT THE STABLE EQUILIBRIUM

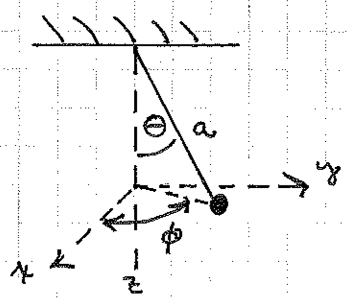
$$K_{EFF} = \left.\frac{d^2 V_{EFF}}{d\Theta^2}\right|_{\Theta=0} = mga - ma^2\Omega^2$$

$$\omega = \sqrt{\frac{K_{EFF}}{M_{EFF}}} = \sqrt{\frac{g}{a} - \Omega^2} = \sqrt{\frac{g}{a}}\sqrt{1 - \frac{a\Omega^2}{g}}$$

IF $\Omega \to 0$, $\omega \to \sqrt{g/a}$ AS FOR A SIMPLE PENDULUM!

EXERCISE: SKETCH THE MOTION OF THE OSCILLATING MASS

EXAMPLE 2: SPHERICAL PENDULUM (L&L PROB 1, P 33)



A MORE INSTRUCTIVE AND LESS ARTIFICIAL EXAMPLE IS THE SPHERICAL PENDULUM. A MASS M ON A STRING OF LENGTH a CAN MOVE OVER THE SURFACE OF A SPHERE. THIS IS EQUIVALENT TO A MASS SLIDING ON THE INSIDE OF A HEMISPHERICAL BOWL; AND ALSO EQUIVALENT TO THE PRECEEDING EXAMPLE WITH Ω NOT FIXED BUT FREE TO VARY.

WE CAN OF COURSE REGARD THIS AS THE SUPERPOSITION OF TWO SIMPLE PENDULA, ONE IN THE X-Z PLANE AND THE OTHER IN THE Y-Z PLANE. FOR SMALL OSCILLATIONS ABOUT THE VERTICAL, THIS VIEW IS QUITE USEFUL.

BUT WE SENSE ANOTHER KIND OF EQUILIBRIUM MOTION. THE PENDULUM SWEEPS OUT A CONE OF CONSTANT ANGLE Θ. (B & O PROB. 16, P. 81). THE GENERAL MOTION CAN BE THOUGHT OF AS A KIND OF OSCILLATION ABOUT THIS CONICAL MOTION. THIS WILL BE OUR APPROACH.

THERE ARE 2 DEGREES OF FREEDOM IN THIS PROBLEM, SO IT'S NOT IMMEDIATELY OBVIOUS HOW OUR 1-DIMENSIONAL METHODS WILL HELP

ENERGY = $T + V$ IS CONSERVED.

IN ADDITION, ANGULAR MOMENTUM ABOUT THE VERTICAL AXIS IS CONSERVED. WE CAN USE THIS TO ELIMINATE $\dot{\phi}$ FROM THE EQUATIONS – AND WE ARE LEFT WITH A 1-DIMENSIONAL PROBLEM!

WE BEGIN BY DERIVING THE EQUATIONS OF MOTION

$$L = T - V = \tfrac{1}{2} m a^2 \dot{\theta}^2 + \tfrac{1}{2} m a^2 \sin^2\theta\, \dot{\phi}^2 - mga(1-\cos\theta)$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{CONST} = m a^2 \sin^2\theta\, \dot{\phi} = \text{ANGULAR MOMENTUM ABOUT } z \text{ AXIS}$$

WE DEFINE $L_0 = m a^2 \sin^2\theta\, \dot{\phi}$ (CAREFUL, $L_0 \neq L$)

[CAUTION: IT IS NOT LEGAL TO SUBSTITUTE THIS INTO THE LAGRANGIAN BEFORE DIFFERENTIATING BY $\dot{\theta}$, θ, ETC.]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m a^2 \ddot{\theta} = \frac{\partial L}{\partial \theta} = m a^2 \sin\theta \cos\theta\, \dot{\phi}^2 - mga \sin\theta$$

$$\text{SO} \quad m a^2 \ddot{\theta} = -mga\sin\theta + \frac{L_0^2 \cos\theta}{m a^2 \sin^3\theta} \equiv -\frac{dV_{EFF}}{d\theta}$$

HENCE $V_{EFF} = mga(1-\cos\theta) + \dfrac{L_0^2}{2 m a^2 \sin^2\theta}$ = EFFECTIVE POTENTIAL

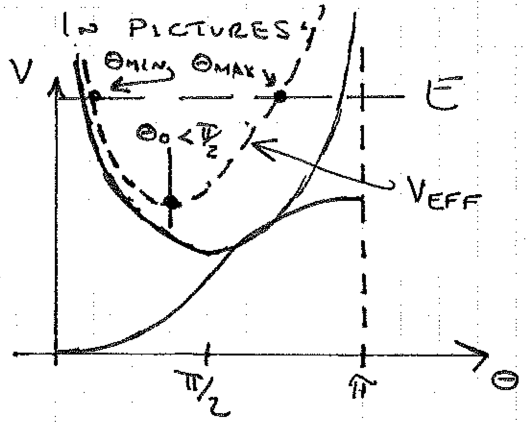
[ALSO, $m_{EFF} = m a^2$]

AT EQUILIBRIUM $\dfrac{dV_{EFF}}{d\theta} = 0 \Rightarrow \dfrac{\cos\theta_0}{\sin^4\theta_0} = \dfrac{m^2 a^3 g}{L_0^2}$

IF $L_0 = 0$ WE HAVE $\theta_0 = 0 \Rightarrow$ SIMPLE PENDULUM MOTION IN A VERTICAL PLANE

FOR $L_0 > 0$ WE HAVE $0 < \theta_0 < \pi/2$.

IN PICTURES:



THE MOTION IS CONFINED TO

$0 < \theta_{MIN} < \theta < \theta_{MAX} < \pi$

THERE IS NO PASSING THRU $\theta = 0$ FOR $L_0 > 0$.

WE COULD ALSO HAVE CONSTRUCTED THE EFFECTIVE POTENTIAL BY EXAMINING THE HAMILTONIAN — IN THIS CASE THE TOTAL ENERGY.

$$H = E = \tfrac{1}{2} m a^2 \dot{\theta}^2 + \tfrac{1}{2} m a^2 \sin^2\theta\, \dot{\phi}^2 + mga(1-\cos\theta)$$

$$= \underbrace{\tfrac{1}{2} m a^2 \dot{\theta}^2}_{T_{EFF}} + \underbrace{\frac{L_0^2}{2ma^2\sin^2\theta} + mga(1-\cos\theta)}_{V_{EFF}}$$

THEN TO FIND THE EQUILIBRIUM POINT WE WOULD SET $\frac{dV_{EFF}}{d\theta} = 0$

FOR CONICAL MOTION EXACTLY AT EQUILIBRIUM ANGLE Θ_0,

$$\frac{\sin^4\Theta_0}{\cos\Theta_0} = \frac{L_0^2}{m^2a^3g} = \frac{a}{g}\sin^4\Theta_0\,\Omega^2 \qquad \Omega = \dot{\phi} = \text{CONSTANT}$$

OR $\Omega = \frac{L_0}{ma^2\sin^2\Theta_0} = \sqrt{\frac{g}{a\cos\Theta_0}}$ IS THE ANGULAR VELOCITY ABOUT z AXIS

FOR OSCILLATIONS ABOUT EQUILIBRIUM, WE NEED K_{EFF}

$$\frac{dV_{EFF}}{d\theta} = mga\sin\theta - \frac{L_0^2\cos\theta}{ma^2\sin^3\theta}$$

$$\frac{d^2V_{EFF}}{d\theta^2} = mga\cos\theta + \frac{L_0^2}{ma^2\sin^2\theta} + \frac{3L_0^2\cos^2\theta}{ma^2\sin^4\theta}$$

AT Θ_0, $K_{EFF} = mga\cos\Theta_0 + mga\frac{\sin^2\Theta_0}{\cos\Theta_0} + 3mga\cos\Theta_0$

$$= \frac{mga}{\cos\Theta_0}\left(\sin^2\Theta_0 + 4\cos^2\Theta_0\right) = mga\frac{(1+3\cos^2\Theta_0)}{\cos\Theta_0}$$

THE <u>OSCILLATION</u> FREQUENCY IS

$$\omega = \sqrt{\frac{K_{EFF}}{M_{EFF}}} = \sqrt{\frac{g}{a\cos\Theta_0}}\sqrt{1+3\cos^2\Theta_0}$$

IF $\Theta \sim \Theta_0$ ALWAYS, THE ROTATIONAL FREQUENCY IS STILL $\Omega \sim \sqrt{\frac{g}{a\cos\Theta_0}}$

$$\text{SO} \quad \omega \sim \Omega\sqrt{1+3\cos^2\Theta_0} \geq \Omega$$

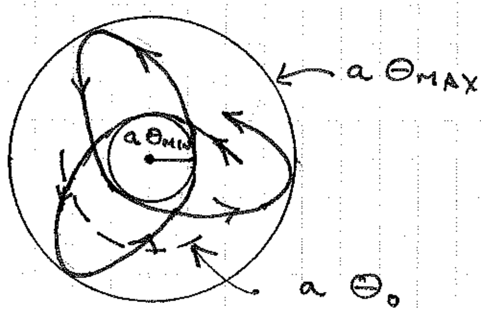
<u>EXERCISE</u>: IF WE WRITE $\Theta = \Theta_0 + \epsilon\sin\omega t$, $\epsilon \ll \Theta_0$, FOR THE SMALL OSCILLATIONS, SHOW THAT $\phi \approx \Omega t + 2\epsilon\frac{\Omega}{\omega}\cot\Theta_0\cos\omega t$

THAT IS, THE MOTION INVOLVES OSCILLATIONS IN BOTH Θ AND ϕ!

THE OSCILLATION FREQUENCY IS HIGHER THAN THE ROTATION FREQUENCY. IN GENERAL THE OSCILLATION ORBIT IS NOT PERIODIC IN SPACE. IF IT WERE, WE MUST HAVE $\frac{\omega}{\Omega} = 1, 2, 3 \ldots$

THIS CAN ONLY HAPPEN AT $\Theta_0 = 0$ OR $90°$ - THE EXTREME CASES DISCUSSED FURTHER BELOW.

ONE WAY TO SKETCH THE ORBIT IS A POLAR COORD PLOT OF $r \equiv a\Theta$, AND ϕ

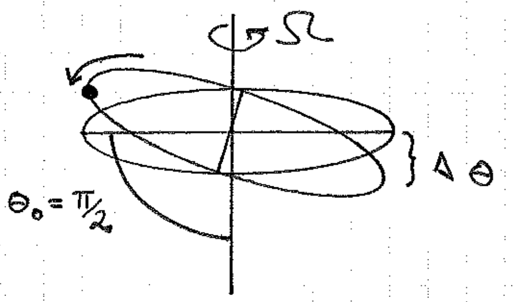


THE ORBIT IS NOT CLOSED. WE CAN SAY THAT THE ORBIT PRECESSES - THE VALUE OF ϕ AT Θ_{MAX} VARIES FROM TURN TO TURN.

WE NOW DISCUSS THE LIMITING CASES OF THE MOTION.

a) $L_0 \to \infty \Rightarrow \Theta_0 \to \pi/2$, $\Omega \to 0$ AND $\omega \to \Omega$

WE HAVE EXACTLY ONE OSCILLATION PER REVOLUTION.



IN THIS CASE THE ORBIT OF AN OSCILLATION ABOUT EQUILIBRIUM IS JUST A TILTED CIRCLE!

IF $L_0 \to \infty$ GRAVITY BECOMES IRRELEVANT, AND WE EXPECT CIRCULAR ORBITS - THRU ANY GREAT CIRCLE.

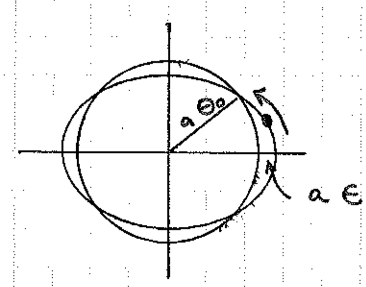
b) $L_0 \to 0$, $\Theta_0 \to 0$, $\Omega \to \sqrt{g/a}$ BUT $\omega \to 2\Omega$

WE MIGHT THINK THAT WE SHOULD BE CONVERGING ON THE CASE OF A SIMPLE PENDULUM - SO $\omega \to 2\Omega$ SEEMS ODD: 2 OSCILLATIONS PER REVOLUTION...

HOWEVER, ONLY FOR $L_0 = 0$ DO WE GET MOTION IN A VERTICAL PLANE. FOR $L_0 \approx \epsilon$ THE EQUILIBRIUM ORBIT IS STILL A TINY CONE.

IF $\omega = 2\Omega$ THE OSCILLATING ORBIT IS ELLIPSE-LIKE:

$$\Theta = \Theta_0 + \epsilon \cos\left(2\sqrt{\frac{g}{a}}\, t\right)$$



CAN WE RECONCILE THIS WITH AN ELEMENTARY ANALYSIS?

WE CONSIDER THE MOTION TO BE THE SUM OF THAT OF AN x-z PENDULUM AND A y-z PENDULUM.

$$\Theta^2 = \Theta_x^2 + \Theta_y^2$$

WHERE
$$\Theta_x = A_x \cos\left(\sqrt{\frac{g}{a}}\, t + \alpha_x\right)$$
$$\Theta_y = A_y \cos\left(\sqrt{g/a}\, t + \alpha_y\right)$$
IN GENERAL

FOR THE EQUILIBRIUM CONICAL MOTION

$$A_x = A_y = \Theta_0 \qquad \alpha_x = 0 \qquad \alpha_y = -\pi/2$$

i.e.
$$\Theta_x = \Theta_0 \cos\left(\sqrt{g/a}\, t\right)$$
$$\Theta_y = \Theta_0 \sin\left(\sqrt{g/a}\, t\right)$$
$$\Theta^2 = \Theta_x^2 + \Theta_y^2 = \Theta_0^2$$

A MOTION WHICH DEVIATES FROM THIS IS

$$\Theta_x = (\Theta_0 + \epsilon) \cos\left(\sqrt{g/a}\, t\right)$$
$$\Theta_y = (\Theta_0 - \epsilon) \sin\left(\sqrt{g/a}\, t\right)$$

WHICH IS AN ELLIPSE $\left(\frac{\Theta_x}{\Theta_0+\epsilon}\right)^2 + \left(\frac{\Theta_y}{\Theta_0-\epsilon}\right)^2 = 1$

(THE OTHER SIMPLE CHOICE: $\Theta_y = (\Theta_0 + \epsilon)$ ....) JUST LEADS TO A NEW CONE, NOT AN OSCILLATION ABOUT THE CONE AT Θ_0)

AND $\theta^2 = \theta_x^2 + \theta_y^2 = \theta_0^2 + \epsilon^2 + 2\epsilon\theta_0\left[\cos^2\left(\sqrt{g/a}\,t\right) - \sin^2\left(\sqrt{g/a}\,t\right)\right]$

$$\sim \theta_0^2 + 2\epsilon\theta_0 \cos\left(2\sqrt{g/a}\,t\right) \qquad [\epsilon^2 \sim 0]$$

$$\theta \sim \theta_0 \sqrt{1 + \frac{2\epsilon}{\theta_0}\cos\left(2\sqrt{g/a}\,t\right)}$$

$$\theta \sim \theta_0 + \epsilon \cos\left(2\sqrt{g/a}\,t\right)$$

AS DERIVED BY OUR OTHER MEANS.

IF YOU WISH YOU COULD CONSIDER ANOTHER KIND OF DEVIATION FROM CONICAL MOTION:

$$\theta_x = \theta_0 \cos\left(\sqrt{g/a}\,t - \epsilon\right)$$

$$\theta_y = \theta_0 \sin\left(\sqrt{g/a}\,t + \epsilon\right)$$

THIS LEADS TO $\theta \sim \theta_0\left(1 + \epsilon \sin\left(2\sqrt{g/a}\,t\right)\right)$

WHICH AGAIN IS AN ELLIPSE-LIKE ORBIT WITH 2 OSCILLATIONS PER REVOLUTION.