Ph 205 LECTURE 6

MORE ABOUT LAGRANGE'S EQUATIONS

CONSERVATION LAWS CONT'D

WE SAW THAT INVARIANCE OF THE LAGRANGIAN IN TIME $\Leftrightarrow$ CONSERVATION OF ENERGY. THIS IS THE FIRST SIGN OF AN IMPORTANT PATTERN: IF THE LAGRANGIAN IS INVARIANT UNDER SOME TRANSFORMATION, THERE IS AN ASSOCIATED CONSERVATION LAW. THIS HAS BEEN PREVIEWED IN OUR DISCUSSIONS OF A SYSTEM OF PARTICLES, pp 15-17. SEE ALSO L&L SEC 6 & 7.

MOMENTUM TO SIMPLIFY THE DISCUSSION WE REVERT TO THE DETAILED DESCRIPTION OF A SYSTEM OF PARTICLES OF MASSES m_i AND POSITIONS $\bar{r}_i$.

SUPPOSE L IS INDEPENDENT OF POSITION.

THEN $\delta L = \sum_i \frac{\partial L}{\partial \bar{r}_i} \cdot \delta \bar{r}_i = 0$ IF ALL $\delta \bar{r}_i = \bar{\epsilon}$ = CONSTANT VECTOR

HENCE $\sum_i \frac{\partial L}{\partial \bar{r}_i} = 0$

BUT $\frac{d}{dt} \frac{\partial L}{\partial \dot{\bar{r}}_i} = \frac{\partial L}{\partial \bar{r}_i}$ ACCORDING TO LAGRANGE'S EQUATIONS

SO $\sum_i \frac{\partial L}{\partial \dot{\bar{r}}_i} = \bar{P}$ = CONSTANT

RECALL THAT $T = \sum_i \frac{1}{2} m_i \dot{\bar{r}}_i^2$ AND $\frac{\partial V}{\partial \dot{\bar{r}}_i} = 0$

SO $\bar{P} = \sum_i m_i \dot{\bar{r}}_i = \sum_i \bar{p}_i$ = TOTAL MOMENTUM

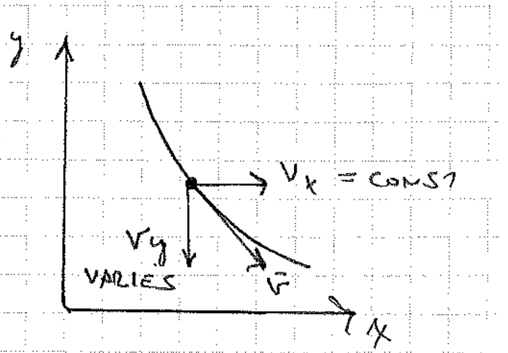
HENCE MOMENTUM IS CONSERVED IF L IS INDEPENDENT OF POSITION.

THIS IS A VECTOR RELATION. IT MAY BE TRUE ALONG ONLY ONE DIRECTION.

EX: CONSIDER A POTENTIAL $V = V(y)$

THEN $\frac{\partial L}{\partial x} = 0 \Rightarrow p_x = mv \sin\theta$ = const

BUT $\frac{\partial L}{\partial y} \neq 0 \Rightarrow p_y$ VARIES



ANGULAR MOMENTUM (L & L sec 9)

SUPPOSE L IS INVARIANT UNDER ROTATIONS. WHAT IS CONSERVED? OF YOUR 3 GUESSES, 2 ARE ALREADY SPOKEN FOR.

RECALL THAT IF A VECTOR $\bar{A}$ IS ROTATING ABOUT AN AXIS WITH ANGULAR VELOCITY $\bar{\omega}$ THEN

$$\frac{d\bar{A}}{dt} = \bar{\omega} \times \bar{A}$$

WE MAKE A SMALL ROTATION $\delta\bar{\theta} = \bar{\omega}\,\delta t$

THEN $\delta\bar{r}_i = \delta\bar{\theta} \times \bar{r}_i \qquad \delta\bar{v}_i = \delta\bar{\theta} \times \bar{v}_i$

AND $$\delta L = \sum_i \left(\frac{\partial L}{\partial \bar{r}_i}\cdot\delta\bar{r}_i + \frac{\partial L}{\partial \dot{\bar{r}}_i}\cdot\delta\bar{v}_i\right) = \sum_i\left(\frac{\partial L}{\partial \bar{r}_i}\cdot\delta\bar{\theta}\times\bar{r}_i + \frac{\partial L}{\partial \bar{v}_i}\cdot\delta\bar{\theta}\times\bar{v}_i\right)$$

$$= \delta\bar{\theta}\cdot\sum_i\left(\bar{r}_i\times\frac{\partial L}{\partial \bar{r}_i} + \bar{v}_i\times\frac{\partial L}{\partial \bar{v}_i}\right) = 0 \quad \text{BY HYPOTHESIS}$$

SINCE $\frac{\partial L}{\partial \bar{v}_i} = m\bar{v}_i$ THE 2ND TERM VANISHES.

ALSO $\frac{\partial L}{\partial \bar{r}_i} = -\frac{\partial V}{\partial \bar{r}_i} = \bar{F}_i = \dot{\bar{p}}_i$

HENCE $\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \frac{d}{dt}\sum_i \bar{r}_i \times \bar{p}_i = 0$

AND $\bar{L} = \sum_i \bar{r}_i \times \bar{p}_i = $ CONSTANT.

THUS INVARIANCE OF THE LAGRANIAN UNDER ROTATIONS $\Leftrightarrow$ CONSERVATION OF ANGULAR MOMENTUM

TIME REVERSAL AND MIRROR REFLECTION

WE HAVE SEEN HOW INVARIANCE OF THE LAGRANGIAN (= INVARIANCE OF THE LAWS OF PHYSICS) UNDER A CONTINUOUS TRANSFORMATION LEADS TO A CONSERVATION LAW:

TIME $\longleftrightarrow$ ENERGY
POSITION $\longleftrightarrow$ MOMENTUM
ROTATION $\longleftrightarrow$ ANGULAR MOMENTUM

(FOR LINGO LOVERS THESE PAIRS ARE SAID TO CONSIST OF CANONICALLY CONJUGATE VARIABLES.)

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THERE EXISTS ANOTHER TYPE OF TRANSFORMATION WHICH LEAVES THE LAGRANGIAN INVARIANT — THE SO-CALLED DISCRETE TRANSFORMATIONS OF TIME REVERSAL AND OF MIRROR REFLECTION (OR MORE TECHNICALLY SPACE INVERSION).

SUPPOSE WE CHANGE t TO $-t$ EVERYWHERE

THEN $\bar{r}(t) \to \bar{r}(-t)$

BUT $\bar{v}(t) \to -v(-t)$ SINCE $dt \to -dt$

$$\bar{a} \to \bar{a}$$
$$m \to m$$
$$\bar{F}(t) \to \bar{F}(-t)$$
$$\bar{p} \to -\bar{p}$$
$$\bar{L} = \bar{r} \times \bar{p} \to -\bar{L}$$
$$T \to T \qquad V \to V \qquad L = T - V \to L$$

AND $\bar{F} = m\bar{a} \quad \to \quad \bar{F}(-t) = m\,\bar{a}(-t)$

SO IF $\bar{r}(t)$ SOLVES A PROBLEM, $\bar{r}(-t)$ IS A SOLUTION TO THE 'TIME REVERSED' PROBLEM, SO LONG AS THE 'INITIAL' CONDITIONS OF VELOCITY ARE ALSO REVERSED.

THAT IS, NEWTON'S LAWS DON'T DISTINGUISH THE DIRECTION OF TIME!

SPACE INVERSION IS THE TRANSFORMATION $\bar{r} \to -\bar{r}$

SO $\bar{v} \to -\bar{v}$, $\bar{a} \to -\bar{a}$, $\bar{p} \to -\bar{p}$, $\bar{L} = \bar{r} \times \bar{p} \to \bar{L}$, $\bar{F} \to -\bar{F}$, $m \to m$

AGAIN $T \to T$, $V \to V$ & $L = T - V \to L$

AND $\bar{F} = m\bar{a}$ REMAINS TRUE UNDER THE TRANSFORMATION.

HENCE NEWTON'S LAWS CAN'T DISTINGUISH BETWEEN OUR UNIVERSE AND THE SPACE INVERTED UNIVERSE!

EXCERCISE: CONVINCE YOURSELF THAT THE TRANSFORMATION OF SPACE INVERSION IS EQUIVALENT TO MIRROR REFLECTION FOLLOWED BY A ROTATION OF 180° ABOUT AN AXIS $\perp$ TO THE MIRROR.

ONCE WE KNOW ABOUT ELECTRICITY, THERE IS A 3RD DISCRETE TRANSFORMATION: CHARGE CONJUGATION

$Q \to -Q$, ALL CHARGES REVERSE SIGN

THEN $\bar{E} \to -\bar{E}$, $\bar{B} \to -\bar{B}$ AND $\bar{F} = Q(\bar{E} + \bar{v} \times \bar{B}) \to \bar{F}$

AND THE LAWS OF MECHANICS REMAIN INVARIANT!

EXCERCISE: WHAT HAPPENS TO $\bar{E}$ AND $\bar{B}$ UNDER TIME REVERSAL AND SPACE INVERSION?

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SO NOW WE HAVE 3 MORE INVARIANCE PRINCIPLES. DO THEY LEAD TO 3 MORE CONSERVATION LAWS? YES – IN A SENSE.

FOR CONTINUOUS TRANSFORMATIONS, A CONTINUOUS VARIABLE TAKES ON A FIXED VALUE IF INVARIANCE HOLDS.

FOR A DISCRETE TRANSFORMATION, A DISCRETE VARIABLE TAKES ON A FIXED VALUE IF INVARIANCE HOLDS. THE DISCRETE VARIABLES ARE CALLED 'QUANTUM NUMBERS' AND PLAY LITTLE ROLE IN CLASSICAL MECHANICS.

HOWEVER, A SERIOUS PROBLEM ARISES. NEWTON'S LAWS DON'T CARE ABOUT TIME REVERSAL, MIRROR REFLECTION, AND CHARGE CONJUGATION. BUT <u>NATURE</u> DOES.

WE ARE CONVINCED THAT TIME BASICALLY RUNS ONE WAY. RUNNING A MOVIE BACKWARDS IS EXACTLY THE TIME REVERSAL TRANSFORMATION, BUT WE JUST LAUGH AT THE RESULT.

MOST PEOPLE ARE RIGHT-HANDED, NOT 50-50 LEFT- AND RIGHT-HANDED. (IS YOUR MIRROR IMAGE LEFT-HANDED OR RIGHT-HANDED?)

MOST NUCLEI ARE POSITIVELY CHARGED, MOST ELECTRONS ARE NEGATIVE. 'ANTI-MATTER' IS FOUND ONLY FOR SHORT TIMES IN HIGH-ENERGY PARTICLE COLLISIONS, AND IN SCIENCE FICTION.

IT APPEARS THAT NATURE DOES NOT FULLY RESPECT THE CONSERVATION LAWS OF DISCRETE TRANSFORMATIONS, WHILE NEWTON'S LAW DO. SOMEHOW NEWTON'S LAWS MUST BE SUPPLEMENTED.

LAGRANGE CERTAINLY DID <u>NOT</u> DO THIS. HIS VIEWPOINT IS EQUIVALENT TO NEWTON'S. SADI CARNOT'S 2ND LAW OF THERMODYNAMICS <u>MAY</u> HAVE A CLUE. HOWEVER THE STATISTICAL MECHANICS EXPLANATION OF THE 2ND LAW IS OF NO FUNDAMENTAL HELP AS IT IS BASED ONLY ON NEWTON'S LAWS: EVEN IF TIME RAN BACKWARDS, THE STATISTICAL MECHANICAL ENTROPY WOULD INCREASE <u>ON THE AVERAGE</u>.

IMAGINE A TIME-REVERSED HOUR GLASS. IF YOU REVERSE THE VELOCITES OF THE SAND <u>RESTING</u> ON THE BOTTOM, WILL THE SAND LEAP UP INTO THE TOP OF THE GLASS? NO! IN THAT WHEN THE SAND IS AT 'REST' ON THE BOTTOM, THE ONLY VELOCITIES ARE MICROSCOPIC THERMAL MOTIONS....

WE LEAVE THESE PUZZLES TO YOUR FUTURE AMUSEMENT.

LAGRANGE'S METHOD FOR NON-INTEGRABLE CONSTRAINTS

SO FAR WE CAN APPLY LAGRANGE'S METHOD ONLY TO SYSTEMS IN WHICH THE CONSTRAINTS ARE EXPRESSED AS

$$g(\bar{r}_1, \ldots \bar{r}_n) = 0 \qquad \text{(HOLONOMIC CONSTRAINTS)}$$

IN 1871 FERRERS GAVE AN EXTENSION OF THE METHOD TO CASES OF NON-INTEGRABLE CONSTRAINTS WHICH, HOWEVER, CAN BE EXPRESSED IN DIFFERENTIAL FORM

$$\sum_k a_{\ell k}\, \delta q_k + b_\ell\, \delta t = 0 \qquad (\ell \text{ SUCH EQUATIONS})$$

OF COURSE, ANY CONSTRAINT OF THE FORM $g(\bar{r}_1 \ldots \bar{r}_n) = 0$ CAN ALSO BE WRITTEN IN THIS FORM — WHICH WILL PROVE USEFUL ON OCCASION.

IT IS USEFUL TO NOTE THAT THE CONSTRAINT EQUATIONS CAN BE PUT IN THE FORM

$$\sum_k a_{\ell k}\, \dot{q}_k + b_\ell = 0$$

THAT IS, THE CONSTRAINT IS A SIMPLE RELATION BETWEEN THE VELOCITIES RATHER THAN THE COORDINATES.

BASICALLY THE METHOD CONSISTS OF PUTTING THE CONSTRAINT FORCES BACK INTO LAGRANGE'S EQUATIONS. IN THIS CASE LAGRANGE'S METHOD LOOKS MORE AND MORE LIKE A STRAIGHTFORWARD APPLICATION OF NEWTON'S LAWS. BUT WE WILL HAVE LEARNED TO FEEL COMFORTABLE IN GENERALISED COORDINATES, AND FURTHERMORE WE ARE REMINDED OF THE NECESSITY OF ESTABLISHING THE CONSTRAINT RELATIONS.

WE RETURN TO OUR ORIGINAL DERIVATION OF LAGRANGE'S EQUATIONS FROM D'ALEMBERT'S PRINCIPLE:

$$\delta W = \sum_i (m_i \bar{a}_i - F_i^e) \cdot \delta \bar{r}_i = 0$$

WHICH WE TRANSFORMED TO GENERALISED COORDINATES AS

$$\delta W = \sum_j \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \delta q_j = 0 \qquad j = 1 \ldots m$$

(FOR CONSERVATIVE FORCES)

IF THE CONSTRAINTS ARE HOLONOMOUS, THE δq_j ARE INDEPENDENT, AND WE GET m SEPARATE EQUATIONS.

NOW HOWEVER, WE HAVE ℓ NON-HOLONOMOUS CONSTRAINT RELATIONS

$$\sum_j a_{kj}\,\delta q_j + b_k\,\delta t = 0 \qquad k = 1 \ldots \ell$$

SO WE CANNOT WRITE DOWN THE CORRECT EQUATIONS OF MOTION IMMEDIATELY.

SKIP

RECALL THAT IN OUR DERIVATION WE CONSIDER ONLY VIRTUAL DISPLACEMENTS AT A FIXED TIME. SO WE SET $\delta t = 0$ AND HAVE $\sum_j a_{kj}\,\delta q_j = 0$ AS OUR CONSTRAINTS ON THE VIRTUAL DISPLACEMENTS. SUPPOSE THAT THERE IS A CONSTRAINT FORCE, F_K, (A NEWTONIAN RATHER THAN A GENERALISED FORCE) ASSOCIATED WITH THE KTH CONSTRAINT. THEN WE CAN INTERPRET THE CONSTRAINT RELATION AS REQUIRING THAT THE VIRTUAL WORK DONE BY THE CONSTRAINT FORCE VANISHES:

$$\delta W_K = F_K\,\delta x_K = F_K \sum_j a_{Kj}\,\delta q_j = 0$$

WE JUSTIFY THIS INTERPRETATION BELOW. THIS POINT OF VIEW MAY HELP YOU IN DERIVING THE CONSTRAINT RELATIONS.

RETURNING TO THE TASK OF FINDING THE EQUATIONS OF MOTION, WE NOTE THAT WE MUST MINIMIZE THE VIRTUAL WORK ($\delta W = 0$ ABOUT A MINIMUM) IN DISPLACEMENTS SUBJECT TO CONSTRAINTS. THIS IS EXACTLY THE SITUATION THAT LAGRANGE MULTIPLIERS WERE INVENTED FOR.

LET λ_K = MULTIPLIER CORRESPONDING TO THE KTH CONSTRAINT.

THEN $$\delta W^* = \sum_j \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} - \sum_K \lambda_K a_{Kj} \right) \delta q_j = 0$$

IS OUR NEW MINIMIZATION CRITERION. NOW WE CAN DEMAND THAT EACH TERM VANISH SEPARATELY. WE THUS OBTAIN THE M EQUATIONS

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_K \lambda_K a_{Kj}}$$

IN M+ℓ UNKNOWNS $\{ q_j(t), \lambda_K(t),\ j = 1 \ldots M,\ K = 1 \ldots \ell \}$

THERE ARE ℓ ADDITIONAL CONSTRAINT EQUATIONS

$$\sum_j a_{Kj}\,\dot q_j + b_K = 0$$

SO A SOLUTION IS POSSIBLE IN PRINCIPLE!

IT IS VERY IMPORTANT TO NOTE THAT THE MULTIPLIERS λ_K ARE FUNCTIONS OF TIME IN GENERAL. THEREFORE IN PRACTICE YOU MUST ELIMINATE THEM FROM THE EQUATIONS OF MOTION BEFORE INTEGRATING (BY USING THE CONSTRAINT EQUATIONS.).

WE RETURN TO THE MEANING OF THE COEFFICIENTS a_{Kj}. SUPPOSE THAT INSTEAD OF TRYING TO DEAL WITH THE NON-HOLONOMOUS CONSTRAINTS VIA THE ABOVE METHOD, WE SIMPLY REGARDED THESE CONSTRAINTS AS DUE TO SOME ADDITIONAL EXTERNAL FORCES. LET Q_j BE THE GENERALISED FORCES CORRESPONDING TO THE NON-HOLONOMOUS CONSTRAINT FORCES. THEN WE COULD HAVE WRITTEN

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$

(THE ORDINARY EXTERNAL FORCES ARE CONTAINED IN THE PIECE $-\frac{\partial V}{\partial q_j}$)

BUT WE JUST DERIVED THAT

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_K \lambda_K a_{Kj}$$

HENCE $Q_j = \sum_K \lambda_K a_{Kj}$ = GENERALISED FORCE DUE TO THE NON-HOLONOMOUS CONSTRAINTS

THE VIRTUAL WORK OF THESE CONSTRAINT FORCES IS

$$\delta W = \sum_j Q_j \delta q_j = \sum_j \sum_K \lambda_K a_{Kj} \delta q_j = \sum_K \lambda_K \sum_j a_{Kj} \delta q_j$$

SO $\delta W_K = \lambda_K \sum_j a_{Kj} \delta q_j$ = WORK OF KTH CONSTRAINT FORCE.

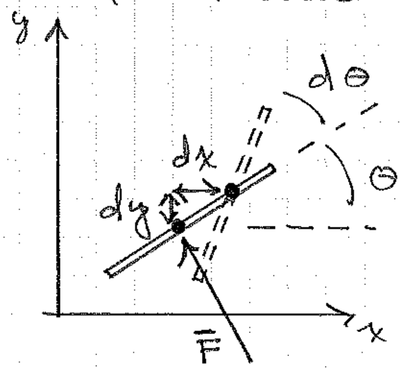
IF $\sum_j a_{Kj} \delta q_j$ HAS THE DIMENSIONS OF LENGTH, THEN λ_K IS AN ORDINARY FORCE. THUS IF WE ARE CAREFUL ABOUT WRITING THE CONSTRAINT EQUATIONS TO HAVE THE PROPER DIMENSIONS (LENGTH), WE WILL AUTOMATICALLY DETERMINE THE CONSTRAINT FORCES λ_K AS PART OF OUR DERIVATION OF THE EQUATIONS OF MOTION.

THIS SPIRIT OF THIS SHOULD BE FAMILIAR FROM YOUR EXPERIENCE WITH ELEMENTARY METHODS: IF YOU CAN'T GUESS A CLEVER SET OF VARIABLES WHICH AVOIDS THE QUESTION OF NORMAL FORCES, TENSIONS IN STRINGS, ETC., YOU ARE GOING TO HAVE TO INCLUDE THESE FORCES EXPLICITLY IN THE EQUATIONS OF MOTION. IN ADDITION, YOU MUST USE THE CONSTRAINT EQUATIONS TO ELIMINATE THE CONSTRAINT FORCES BEFORE YOU ARRIVE AT THE DESIRED EQUATIONS OF MOTION...

LAGRANGE'S METHOD IS JUST A SYSTEMATIC WAY OF ORGANISING THESE 'INTUITIVE' APPROACHES. IN REAL LIFE IT APPEARS JUST AS WELL TO STICK TO THE ELEMENTARY APPROACH IN PROBLEMS WITH NON-HOLONOMIC CONSTRAINTS!

EXAMPLE. THE ONE-LEGGED ICE SKATER

AN ICE SKATE IS APPARENTLY THE SIMPLEST EXAMPLE OF A NON-HOLONOMOUS SYSTEM — IN THAT ONLY 3 COORDINATES ARE INVOLVED: x, y & θ



THE ICE SKATE IS CONSTRAINED NOT TO SLIP SIDEWAYS BY THE CONSTRAINT FORCE $\bar{F}(t)$. BUT THE SKATE CAN SLIDE ALONG ITS DIRECTION OF MOTION, AND TURN ABOUT THE POINT OF APPLICATION OF THE FORCE.

A SIMPLIFICATION OCCURS IF $\bar{F}$ IS APPLIED AT THE AXIS OF THE C.M.

WE CAN SOLVE THIS PROBLEM BY ELEMENTARY MEANS.

$\bar{F}$ EXERTS NO TORQUE ABOUT THE C.M. $\Rightarrow \bar{L}_{CM} = \text{CONST} \Rightarrow \omega = \dot{\theta} = \text{CONST}.$

ALSO $\bar{F}\cdot\bar{v}_{CM} = 0 \Rightarrow \dfrac{dKE}{dt} = 0 \Rightarrow |\bar{v}| = \text{CONST} = v$ (SINCE $\frac{1}{2}I\dot{\theta}^2 = \text{CONST}$)

SO THE MOTION HAS CONSTANT VELOCITY & A CONSTANT RATE OF TURNING $\Rightarrow$ A CIRCLE

NOW $F = \dfrac{mv^2}{\rho}$ WHERE $\rho = \dfrac{dl}{d\theta} = \dfrac{dl}{dt}\Big/\dfrac{d\theta}{dt} = \dfrac{v}{\omega}$ = RADIUS OF CURVATURE

SO $F = mv\omega$ THAT'S IT!

WE NOW ILLUSTRATE LAGRANGE'S METHOD. THE VARIABLES ARE x, y & θ.

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 \qquad V = 0$$

THE CONSTRAINT IS JUST $dy/dx = \tan\theta$

BUT WE SHOULD WRITE THE CONSTRAINT SO THAT EACH TERM HAS THE DIMENSIONS OF LENGTH:

$$\cos\theta\, dy - \sin\theta\, dx \quad \text{[BETTER THAN } dy - \tan\theta\, dx = 0 \text{ TRY IT!]}$$

OR $\cos\theta\, \dot{y} - \sin\theta\, \dot{x} = 0$

THE CONSTRAINT COEFFICIENTS ARE THEN $a_x = -\sin\theta$, $a_y = \cos\theta$, $a_\theta = 0$

THE EQUATIONS OF MOTION ARE [$\lambda \equiv$ MULTIPLIER]

$$x: \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial T}{\partial x} = 0 \quad \Rightarrow \quad m\ddot{x} = -\lambda \sin\theta$$

$$y: \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{y}} = m\ddot{y} \qquad \frac{\partial T}{\partial y} = 0 \quad \Rightarrow \quad m\ddot{y} = \lambda \cos\theta$$

$$\theta: \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = I\ddot{\theta} \qquad \frac{\partial T}{\partial \theta} = 0 \quad \Rightarrow \quad I\ddot{\theta} = 0$$

SO LAGRANGE'S METHOD HAS DUPLICATED THE 'OBVIOUS' RESULTS OF THE ELEMENTARY METHOD, IF WE IDENTIFY $\lambda = F$!

CLEARLY $\dot{\theta} = \omega = \text{CONST}$

SINCE λ IS UNKNOWN, WE MUST COMBINE THE x & y EQUATIONS

$$m\ddot{x}\cos\theta + m\ddot{y}\sin\theta = 0 \quad \text{(THIS CAN BE INTEGRATED IN PRINCIPLE)}$$

$$-m\ddot{x}\sin\theta + m\ddot{y}\cos\theta = \lambda \quad \text{(THIS WILL GIVE } \lambda \text{ EVENTUALLY)}$$

A USEFUL TRICK IS TO TRANSFORM $\dot{x}$ & $\dot{y}$ TO v AND θ BY $\dot{x} = v\cos\theta$, $\dot{y} = v\sin\theta$, WHICH AUTOMATICALLY SATISFIES THE CONSTRAINT EQUATION. THEN

$$\ddot{x} = \dot{v}\cos\theta - v\sin\theta\,\dot{\theta} \qquad \ddot{y} = \dot{v}\sin\theta + v\cos\theta\,\dot{\theta}$$

SO $m\ddot{x}\cos\theta + m\ddot{y}\sin\theta = m\dot{v} = 0 \Rightarrow$ $\underline{v = \text{CONSTANT}}$

FINALLY $\underline{\lambda} = -m\ddot{x}\sin\theta + m\ddot{y}\cos\theta = mv\dot{\theta} = \underline{mv\omega}$

IF YOU WISH, YOU CAN INTEGRATE

$$\dot{x} = v\cos\theta = v\cos(\omega t + \theta_0)$$

$$\dot{y} = v\sin\theta = v\sin(\omega t + \theta_0)$$

TO SHOW $(x - x_0)^2 + (y - y_0)^2 = v^2/\omega^2$

WE LEAVE TO THE HOMEWORK SET THE MUCH MORE COMPLICATED CASE WHERE $\vec{F}$ IS NOT APPLIED AT THE C.M.

HOLONOMIC CONSTRAINT FORCES

WE NOW CAN SEE A PROCEDURE TO 'UNDO' LAGRANGE'S METHOD TO DERIVE HOLONOMIC CONSTRAINT FORCES. IN PRACTICE IT'S PROBABLY EASIER TO GET THEM DIRECTLY FROM NEWTON'S LAWS.

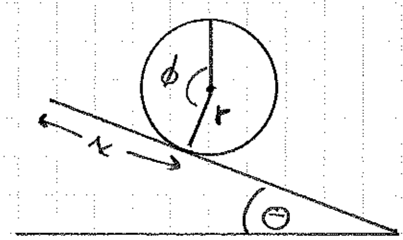
A CONSTRAINT $g(\bar{r}_1 \cdots \bar{r}_n) = 0$

CAN BE WRITTEN $\sum_j \frac{\partial g}{\partial q_j} \delta q_j = 0$ ALSO

SO WE USE THE METHOD OF LAGRANGE MULTIPLIERS WITH

$a_j = \partial g / \partial q_j$ AND λ = DESIRED CONSTRAINT FORCE

EXAMPLE A HOOP ROLLS WITHOUT SLIPPING DOWN AN INCLINED PLANE. WHAT IS THE FORCE OF FRICTION WHICH KEEPS THE HOOP FROM SLIPPING?



AS THE COORDINATES, WE CHOOSE x = DISTANCE ALONG THE INCLINE, AND ϕ = ANGLE OF ROTATION OF THE HOOP.

THE CONSTRAINT IS $x = r\phi$

OR $dx - r\,d\phi = 0$ OR $\dot{x} - r\dot{\phi} = 0$

$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\phi}^2 = \frac{1}{2} m (\dot{x}^2 + r^2 \dot{\phi}^2)$ $\quad V = -mg\,x \sin\theta$

THE CONSTRAINT COEFFICIENTS ARE $a_x = 1$ $\quad a_\phi = -r$

x: $\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = m\ddot{x}$ $\quad \frac{\partial L}{\partial x} = mg\sin\theta \Rightarrow m\ddot{x} - mg\sin\theta = \lambda$

ϕ: $\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = mr^2\ddot{\phi}$ $\quad \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad mr^2\ddot{\phi} = -r\lambda$

NOW $\dot{x} = r\dot{\phi} \Rightarrow \ddot{x} = r\ddot{\phi} \Rightarrow mr\ddot{\phi} - mg\sin\theta = \lambda = -mr\ddot{\phi}$

SO $\ddot{\phi} = \frac{g\sin\theta}{2r}$ $\quad \ddot{x} = \frac{g\sin\theta}{2}$

AND $\lambda = -mr\ddot{\phi} = -\frac{mg}{2}\sin\theta = F_{FRICTION}$. IT POINTS UPHILL