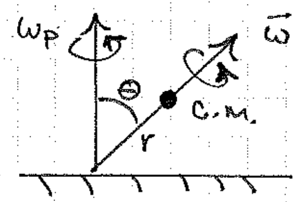
Ph 205 LECTURE 19

MOTION OF A SPINNING TOP INCLUDING GRAVITY

WE WISH TO UNDERSTAND THE MOTION OF A TOP SPINNING WITH ONE POINT FIXED (THE SUPPORT POINT), UNDER THE INFLUENCE OF GRAVITY.

WE HAVE ALREADY SEEN THAT THE IDEALIZED SOLUTION FOR A RAPIDLY SPINNING TOP IS PRECESSION ABOUT THE VERTICAL.

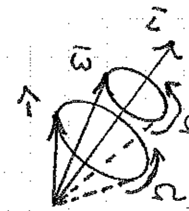
$$\frac{d\bar{L}}{dt} = \bar{\omega}_P \times \bar{L} \quad , \quad \bar{\omega}_P = \frac{Mgr}{I_1 \omega_1}\hat{z}$$



WE HAVE ALSO SEEN THAT IF THERE WERE NO GRAVITY, SO $\bar{N} = 0$, THEN $\bar{L}$ = CONSTANT, AND $\bar{\omega}$ PRECESSES ABOUT $\bar{L}$ (IF $\bar{\omega}$ NOT $\parallel$ TO $\bar{L}$)

$$\frac{d\bar{\omega}}{dt} = \bar{\Omega} \times \omega \qquad \bar{\Omega} = \bar{L}/I_2$$

ALSO THE SYMMETRY AXIS $\hat{1}$ PRECESSES ABOUT $\bar{L}$: $\frac{d\hat{1}}{dt} = \bar{\Omega} \times \hat{1}$



AS A GUESS TO THE GENERAL SOLUTION TO THE CASE INCLUDING GRAVITY, WE MIGHT EXPECT A COMBINATION OF THE TWO MOTIONS. (THE DIFFERENTIAL EQUATION IS LINEAR...)

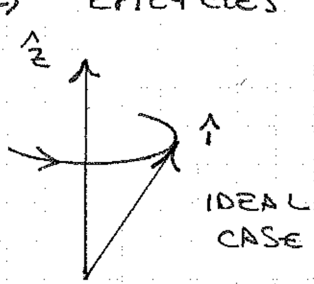
SUPPOSE $\bar{\omega} \neq \bar{L}$, BUT THEY ARE ALMOST PARALLEL, AND BOTH ALMOST ALONG THE SYMMETRY AXIS: $\bar{L} \sim I_1 \omega_1 \hat{1}$

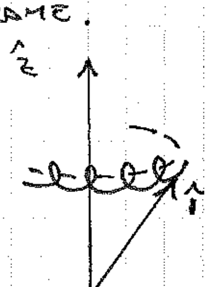
THE 'FREE PRECESSION' $\Omega \sim \frac{I_1 \omega_1}{I_2}$ IS VERY RAPID

THE GYROSCOPIC PRECESSION $\omega_P \sim \frac{Mgr}{I_1 \omega_1}$ IS SLOW.

HENCE ON THE AVERAGE THE RAPID FREE PRECESSION CAN BE IGNORED, AND WE MAINLY NOTICE THE GYROSCOPIC EFFECT.

LOOKING IN MORE DETAIL WE SEE THAT THE SYMMETRY AXIS MOVES IN SMALL CIRCLES ABOUT ITS IDEALIZED POSITION (BODY FRAME)
$\Rightarrow$ 'EPICYCLES' IN THE LAB FRAME.





THE WOBBLING OF THE AXIS IS CALLED <u>NUTATION</u>

Ph 205 LECTURE 19

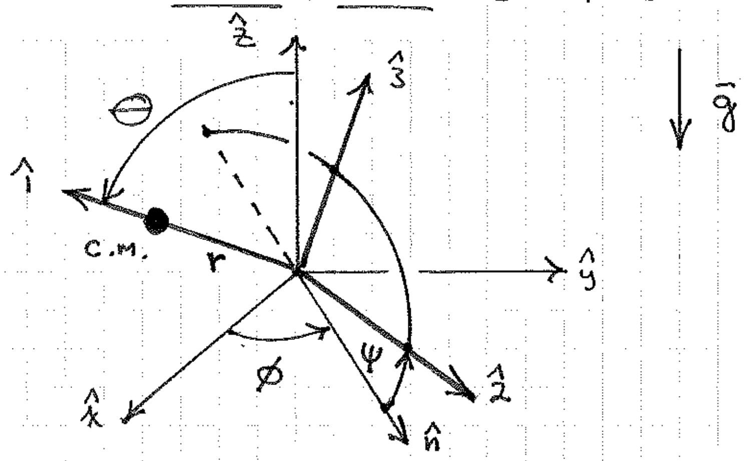
EULER'S ANGLES AND THE EQUATIONS OF MOTION

WE WISH TO DESCRIBE THE PATH OF THE SYMMETRY AXIS IN DETAIL, IN THE LAB FRAME. THE C.M. MOVES BOTH VERTICALLY AND HORIZONTALLY — THE MOTION IS COMPLEX! THE FORCE ON THE SUPPORT POINT IS UNKNOWN — AND MUST BE VARIABLE IF THE C.M. RISES AND FALLS.

WE NEED A METHOD WHICH AVOIDS, IF POSSIBLE, DIRECT USE OF THE UNKNOWN SUPPORT FORCE. THIS IS A GOOD CASE FOR LAGRANGE'S METHOD!

A RIGID BODY WITH ONE POINT FIXED HAS 3 DEGREES OF FREEDOM – 3 ANGULAR COORDINATES.

WE WILL USE THE EULER ANGLES AS DEFINED BY L&L SEC 35.



$\hat{x}, \hat{y}, \hat{z}$ = INERTIAL SPACE AXES

$\hat{1}, \hat{2}, \hat{3}$ = ROTATING BODY AXES $\qquad \hat{1}$ = SYMMETRY AXIS.

THE $\hat{2}$-$\hat{3}$ PLANE INTERSECTS THE $\hat{x}$-$\hat{y}$ PLANE ALONG LINE $\hat{n}$, WHERE $\hat{n} \sim \hat{z} \times \hat{1}$ DEFINES THE SIGN OF $\hat{n}$.

THE 3 EULER ANGLES ARE

Θ = ANGLE BETWEEN $\hat{1}$ & $\hat{z}$

ϕ = ANGLE BETWEEN $\hat{x}$ & $\hat{n}$

ψ = ANGLE BETWEEN $\hat{n}$ & $\hat{2}$

IN GENERAL, ALL 3 ANGLES VARY WITH TIME.

$\dot{\psi}$ = ANGULAR VELOCITY ABOUT THE SYMMETRY AXIS (IF Θ, ϕ CONSTANT)

$\dot{\phi}$ = PRECESSION RATE ABOUT THE VERTICAL

$\dot{\Theta}$ = NUTATION RATE – THE NEW MOTION WE WISH TO UNDERSTAND.

WE CAN NOW CONSTRUCT THE LAGRANGIAN AND FIND THE EQUATIONS OF MOTION.

WE WILL IGNORE THE FACT THAT THE EARTH IS A ROTATING FRAME, AND NEGLECT THE CORRESPONDING CORIOLIS & CENTRIFUGAL FORCES — WHICH ARE IMPORTANT FOR THE GYRO COMPASS.

$$L = T - V = T - mgr\cos\theta$$

WE USE THE SUPPORT POINT, NOT THE C.M., AS OUR ORIGIN. AS THIS POINT IS AT REST, WE CAN SIMPLY WRITE $T = \frac{1}{2}\bar{\omega}\cdot\bar{\bar{I}}_p\cdot\bar{\omega}$ AND $\bar{L} = \bar{\bar{I}}_p\cdot\bar{\omega}$, WHERE $\bar{\bar{I}}_p$ IS CALCULATED ABOUT THE SUPPORT POINT. THEN ALONG THE PRINCIPAL AXES, $I_2 = I_3 = I_{2\,cm} + Mr^2$, AND $I_1 = I_{1\,cm}$.

AND $$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\left(\omega_2^2 + \omega_3^2\right)\right)$$

BUT WE NOW NEED COMPONENTS OF THE TOTAL ANGULAR VELOCITY ALONG THE BODY AXES. THIS IS A BIT TRICKY.

$$\bar{\omega} = \dot{\psi}\,\hat{1} + \dot{\phi}\,\hat{z} + \dot{\theta}\,\hat{n}$$

THUS WE NEED $\hat{z}$ AND $\hat{n}$ IN THE BODY FRAME

$$\hat{z} = \hat{1}\cos\theta + \hat{2}\sin\theta\sin\psi + \hat{3}\sin\theta\cos\psi$$

$$\hat{n} = \hat{2}\cos\psi - \hat{3}\sin\psi$$ (IS IN THE $\hat{2}$-$\hat{3}$ PLANE)

HENCE
$$\omega_1 = \dot{\psi} + \dot{\phi}\cos\theta \qquad (\neq \dot{\psi} \text{ ONLY})$$
$$\omega_2 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_3 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

AND $$T = \frac{1}{2}\left[I_1\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + I_2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)\right]$$

<u>EXAMPLE</u>: <u>FREE</u> <u>PRECESSION</u> $\Longleftrightarrow$ $g = 0$ & C.M. FIXED

THEN $\bar{N} = 0 \Rightarrow \bar{L} = \text{const}$. <u>CHOOSE</u> z AXIS ALONG $\bar{L}$

THEN
$$L_1 = L\cos\theta = I_1\omega_1 = I_1(\dot{\psi} + \dot{\phi}\cos\theta)$$
$$L_2 = L\sin\theta\sin\psi = I_2\omega_2 = I_2(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)$$
$$L_3 = L\sin\theta\cos\psi = I_3\omega_3 = I_2(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)$$

WHILE $\bar{L} = \text{const.}$ IN THE LAB FRAME, THE COMPONENTS OF $\bar{L}$ IN THE BODY FRAME, (L_1, L_2, L_3) ARE NOT NECESSARILY CONSTANT!

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COMBINING THE 2ND & 3RD EQUATIONS:

$$\Rightarrow \dot{\theta} = 0, \quad \theta = \text{CONST} \Rightarrow \omega_1 = \frac{L}{I_1}\cos\theta = \text{CONST.}$$

ALSO WE GET $\dot{\phi} = \frac{L}{I_2} = \text{CONST}$ ⇒ PRECESSION RATE OF $\hat{1}$ ABOUT $\bar{L}$

FINALLY, EQ'N #1 ⇒ $\dot{\psi} = L\,\frac{I_2 - I_1}{I_1 I_2}\cos\theta$ FOR WHAT IT'S WORTH.

WE COULD NOW DIFFERENTIATE $L = T - V$ TO OBTAIN THE EQUATIONS OF MOTION. BUT FIRST LET'S LOOK FOR CONSERVED QUANTITIES.

a) $\partial L/\partial t = 0 \Rightarrow E$ IS CONSERVED.

$$E = \frac{I_1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{I_2}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgr\cos\theta$$

b) $\partial L/\partial \phi = 0 \Rightarrow P_\phi = \text{CONST}$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta) + I_2\dot{\phi}\sin^2\theta \equiv \underline{L_z}$$

$$= I_1\dot{\psi}\cos\theta + \dot{\phi}(I_1\cos^2\theta + I_2\sin^2\theta)$$

c) $\partial L/\partial \psi = 0 \Rightarrow P_\psi = \text{CONST.}$

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_1(\dot{\psi} + \dot{\phi}\cos\theta) \equiv \underline{L_1}$$

SINCE TORQUE $\bar{N} = \bar{r} \times m\bar{g} \sim \hat{n}$, WE EXPECT COMPONENTS OF $\bar{L}$ ALONG AXES $\perp$ TO $\hat{n}$ TO BE CONSERVED.

INSTEAD OF USING $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$ TO GET THE 3RD EQUATION OF MOTION (A TORQUE EQUATION), WE CAN USE THE CONSERVATION LAWS b) & c) TO ELIMINATE $\dot{\phi}$ & $\dot{\psi}$ IN THE ENERGY EXPRESSION — AND REDUCE THE PROBLEM TO ONE DIMENSION!

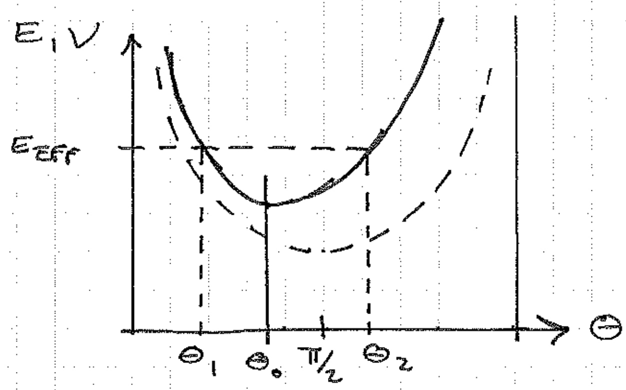
c) $\Rightarrow \dot{\psi} + \dot{\phi}\cos\theta = L_1/I_1$

b) $\Rightarrow \dot{\phi} = \dfrac{L_z - L_1\cos\theta}{I_2\sin^2\theta}$

a) $\Rightarrow$ $$E = \frac{I_1}{2}\left(\frac{L_1}{I_1}\right)^2 + \frac{I_2}{2}\dot{\theta}^2 + \frac{I_2}{2}\frac{(L_z - L_1\cos\theta)^2}{I_2^2\sin^4\theta}\sin^2\theta + mgr\cos\theta$$

$$\underbrace{E - \frac{L_1^2}{2I_1}}_{E_{EFF}} = \underbrace{\frac{1}{2}I_2\dot{\theta}^2}_{T_{EFF}} + \underbrace{mgr\cos\theta + \frac{(L_z - L_1\cos\theta)^2}{2I_2\sin^2\theta}}_{V_{EFF}}$$

THE MOTION OF θ IS NOW REDUCED TO A ONE DIMENSIONAL PROBLEM, WITH A 'FICTITIOUS FORCE' DERIVABLE FROM THE EFFECTIVE POTENTIAL $V_{EFF}(\theta)$.



IN GENERAL V_{EFF} HAS A MINIMUM AT θ_0, WHICH CORRESPONDS TO STEADY MOTION AT FIXED θ $\Leftrightarrow$ SIMPLE PRECESSION.

IF $E_{EFF} > V_{EFF}(\theta_0)$, THEN θ OSCILLATES BACK AND FORTH BETWEEN θ_1 AND θ_2 – THIS IS THE <u>NUTATION</u>. MEANWHILE THE PRECESSION RATE $\dot{\phi} = \dfrac{L_z - L_1\cos\theta}{I_2\sin^2\theta}$ VARIES SINCE θ VARIES.

1) <u>STEADY PRECESSION</u> $\theta = \theta_0$

TO FIND θ_0, WE SET $V'_{EFF} = 0$

$$V'_{EFF} = \frac{(L_z - L_1\cos\theta)L_1}{I_2\sin\theta} - \frac{(L_z - L_1\cos\theta)^2\cos\theta}{I_2\sin^3\theta} - mgr\sin\theta$$

$$= \frac{(L_z - L_1\cos\theta)(L_1 - L_z\cos\theta)}{I_2\sin^3\theta} - mgr\sin\theta$$

SO $$mgrI_2\sin^4\theta_0 = (L_z - L_1\cos\theta_0)(L_1 - L_z\cos\theta_0)$$

WE ALSO WISH THE STEADY PRECESSION RATE $\dot{\phi}_0$

A USEFUL TRICK IS TO REINTRODUCE $\dot{\phi} = \dfrac{L_z - L_1 \cos\theta}{I_2 \sin^2\theta}$

INTO THE EQUATION $V'_{EFF} = 0$. WE NOTE THAT

$$V'_{EFF} = \frac{(L_z - L_1\cos\theta)}{I_2 \sin^3\theta}\left(L_1 \sin^2\theta - (L_z - L_1\cos\theta)\cos\theta\right) - mgr\sin\theta$$

$$= \frac{\dot{\phi}}{\sin\theta}\left(L_1 \sin^2\theta - I_2 \sin^2\theta\cos\theta\,\dot{\phi}\right) - mgr\sin\theta$$

$$= -\sin\theta\left(I_2\cos\theta\,\dot{\phi}^2 - I_1\omega_1\dot{\phi} + mgr\right) \qquad \text{USING } L_1 = I_1\omega_1$$

AT $\theta = \theta_0$, $V'_{EFF} = 0$, SO

$$I_2 \cos\theta\,\dot{\phi}_0^2 - I_1\omega_1\dot{\phi}_0 + mgr = 0$$

$$\dot{\phi}_0 = \frac{I_1\omega_1 \pm \sqrt{I_1^2\omega_1^2 - 4mgrI_2\cos\theta_0}}{2I_2\cos\theta_0}$$

$$= \frac{I_1\omega_1}{2I_2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4mgrI_2\cos\theta_0}{I_1^2\omega_1^2}}\right)$$

FOR $\theta_0 < \pi/2$ — THE CASE OF AN ORDINARY TOP — WE CAN HAVE STEADY PRECESSION ONLY IF $\sqrt{}$ IS REAL:

$$\text{i.e.} \qquad \underline{\omega_1 > \frac{2}{I_1}\sqrt{mgrI_2\cos\theta_0}}$$

THAT IS, IF ω_1 IS TOO SMALL, THE EQUILIBRIUM AT θ_0 IS UNSTABLE & THE TOP FALLS OVER!

SUPPOSE ω_1 IS BIG. THEN

$$\dot{\phi}_0 \sim \frac{I_1\omega_1}{2I_2\cos\theta_0}\left[1 \pm \left(1 - \frac{2mgrI_2\cos\theta_0}{I_1^2\omega_1^2}\right)\right]$$

SURPRISINGLY, THERE ARE 2 ROOTS:

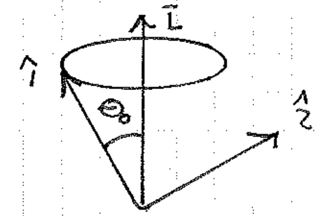
a) $-$ ROOT : $\dot{\phi}_0 \sim \dfrac{mgr}{I_1 \omega_1} \Rightarrow$ SIMPLE PRECESSION, OR 'SLOW PRECESSION'

b) $+$ ROOT : $\dot{\phi}_0 \sim \dfrac{I_1 \omega_1}{I_2 \cos\theta_0} \Rightarrow$ 'FAST PRECESSION'

NOTE THAT GRAVITY PLAYS NO ROLE IN 'FAST PRECESSION'. IT IS ESSENTIALLY TORQUE-FREE PRECESSION ABOUT $\bar{L}$ 'VERTICAL'

$L_1 = I_1 \omega_1 = L \cos\theta_0$

SO $\dot{\phi}_0 \approx \dfrac{L}{I_2}$ EXACTLY AS BEFORE FOR FREE PRECESSION



WE DEMONSTRATE 'FAST' & 'SLOW' PRECESSION WITH A CHILD'S TOY.

IN THE FOLLOWING WE CONCENTRATE ON SLOW PRECESSION.

2) Small Deviations From Steady Motion

FOR θ NEAR θ_0 WE EXPECT SMALL OSCILLATIONS IN θ. BUT IF θ IS NOT CONSTANT, NEITHER IS $\dot{\phi}$, SO WE MIGHT EXPECT OSCILLATIONS IN $\dot{\phi}$ AS WELL.

NOTE: WE COULD, OF COURSE, FIRST FIND $\Omega = \sqrt{\dfrac{V''_{EFF}(\theta_0)}{I_2}}$ AND THEN RETURN TO CONSIDER $\dot{\phi}(t)$

FROM $E_{EFF} = \frac{1}{2} I_2 \dot{\theta}^2 + V_{EFF}$ WE GET

$$I_2 \ddot{\theta} = -V'_{EFF} = \sin\theta \left(I_2 \cos\theta\, \dot{\phi}^2 - I_1 \omega_1 \dot{\phi} + mgr \right)$$

ALSO $I_2 \sin^2\theta\, \dot{\phi} = L_z - I_1 \omega_1 \cos\theta$ $\left[\text{FROM } \dot{\phi} = \dfrac{L_z - L_1 \cos\theta}{I_2 \sin^2\theta} \right]$

SO $I_2 \sin^2\theta\, \ddot{\phi} = -2 I_2 \sin\theta \cos\theta\, \dot{\theta} \dot{\phi} + I_1 \omega_1 \sin\theta\, \dot{\theta}$

WE ARE INTERESTED IN THE CASE OF SLOW PRECESSION AND SMALL OSCILLATIONS $\Rightarrow \dot{\theta}, \dot{\phi}$ SMALL, SO WE CAN NEGLECT QUADRATIC TERMS:

$$I_2 \ddot{\theta} = -I_1 \omega_1 \sin\theta\, \dot{\phi} + mgr \sin\theta$$

$$I_2 \sin\theta\, \ddot{\phi} = I_1 \omega_1 \dot{\theta}$$

WHICH WE REGARD AS THE COUPLED-OSCILLATOR EQUATIONS.

OUR TRIAL SOLUTIONS ARE

$$\theta = \theta_0 + \epsilon \sin \Omega t$$

$$\dot{\phi} = \omega_p + \delta \sin \Omega t$$

WHERE WE EXPECT $\omega_p = \dfrac{mgr}{I_1 \omega_1}$ = SLOW PRECESSION RATE

THEN $\sin\theta \sim \sin\theta_0 + \epsilon \cos\theta_0 \sin\Omega t$, ETC.

PLUGGING INTO THE $\ddot{\theta}$ EQUATION,

$$-I_2 \epsilon \Omega^2 \sin \Omega t = -I_1 \omega_1 (\sin\theta_0 + \epsilon \cos\theta_0 \sin \Omega t)(\omega_p + \delta \sin \Omega t)$$
$$+ mgr(\sin\theta_0 + \epsilon\cos\theta_0 \sin\Omega t)$$

COLLECTING THE CONSTANT TERMS: $0 = (-I_1 \omega_1 \omega_p + mgr)\sin\theta_0$

$$\Rightarrow \underline{\omega_p = \frac{mgr}{I_1\omega_1}} \quad \text{AS EXPECTED.}$$

COLLECTING THE TERMS IN $\sin \Omega t$:

$$-I_2 \epsilon \Omega^2 = -I_1 \omega_1 (\epsilon \omega_p \cos\theta_0 + \delta \sin\theta_0) + \epsilon mgr \cos\theta_0$$
$$= -I_1 \omega_1 \delta \sin\theta_0$$

$$\text{SO} \quad \Omega^2 = \frac{I_1 \omega_1}{I_2} \frac{\delta \sin\theta_0}{\epsilon}$$

TO RELATE δ TO ϵ, USE THE $\ddot{\phi}$ EQUATION:

$$I_2(\sin\theta_0 + \epsilon\cos\theta_0 \sin\Omega t)(\delta \Omega \cos\Omega t) = I_1 \omega_1 \epsilon \Omega \cos \Omega t$$

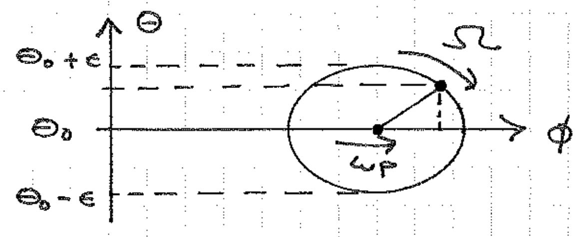
COLLECTING TERMS IN $\cos\Omega t$: $I_2 \delta \Omega \sin\theta_0 = I_1 \omega_1 \epsilon \Omega$

$$\text{SO} \quad \underline{\delta = \frac{I_1\omega_1}{I_2} \frac{\epsilon}{\sin\theta_0}}$$

$\Rightarrow \Omega^2 = \left(\dfrac{I_1\omega_1}{I_2}\right)^2$ OR $\underline{\Omega = \dfrac{I_1 \omega_1}{I_2}}$ = 'FREE PRECESSION' RATE AS ON P. 200

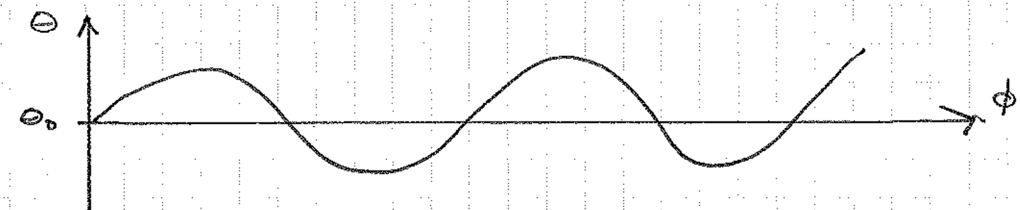
ALSO $\delta = \dfrac{\epsilon \Omega}{\sin\theta_0}$.

Thus
$$\theta = \theta_0 + \epsilon \sin \Omega t \qquad \omega_p = \frac{mgr}{I_1 \omega_1}$$
$$\dot{\phi} = \omega_p + \frac{\epsilon \Omega}{\sin \theta_0} \sin \Omega t$$
$$\phi = \omega_p t - \frac{\epsilon \cos \Omega t}{\sin \theta_0} \qquad \Omega = \frac{I_1 \omega_1}{I_2}$$

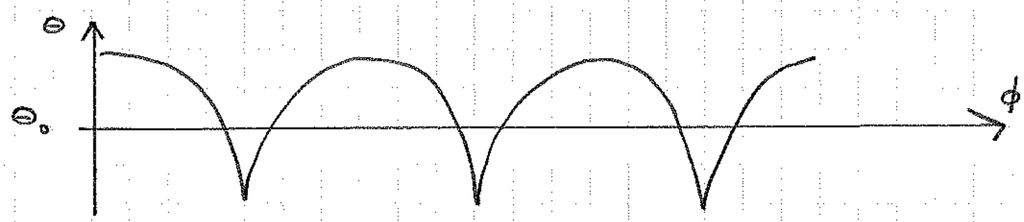


In θ–ϕ space, the motion is like a distorted wheel rolling & slipping.

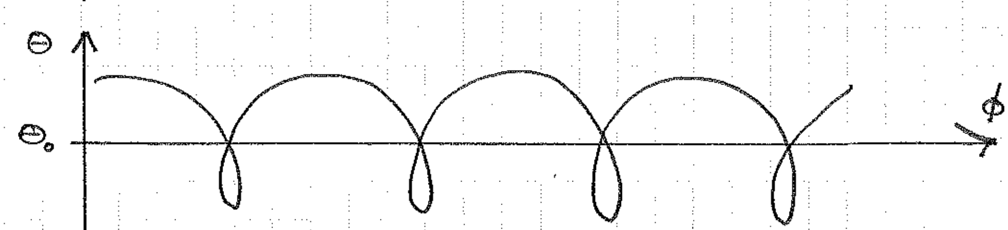
$\omega_p > \Omega$



$\omega_p = \Omega$



$\omega_p < \Omega$
TYPICAL



[Note: small $\theta \Leftrightarrow$ axis near to vertical]

In all cases the motion apears to be steady precession — on the AVERAGE.

Suppose you set a top spinning while holding the axis fixed — then let it go. What happens? Certainly we expect the c.m. to fall at first — but then the torque due to gravity starts to precess the top sideways. The initial falling excites the NUTATION, which continues forever if there is no damping.

In the sketch above, when you let the top go, $\theta = \theta_0 - \epsilon$ $\Leftrightarrow$ height is a maximum.

3) SLEEPING

IF THE TOP IS STARTED WITH THE SYMMETRY AXIS VERTICAL, WILL THE AXIS REMAIN VERTICAL? I.E. WILL THE TOP 'SLEEP'?

FROM OUR EFFECTIVE POTENTIAL, WE SEE THAT $\theta_0 = 0$ IS INDEED A POSSIBLE EQUILIBRIUM POINT:

$$mgr\, I_2 \sin^4\theta_0 = (L_z - L_1\cos\theta_0)(L_1 - L_z\cos\theta_0) \qquad \text{[BOTTOM p. 204]}$$

WHEN $\theta_0 = 0$, $L_1 = L_z$, SO $V'_{EFF} = 0$

OUR GENERAL CONDITION FOR STEADY MOTION IS

$$\omega_1 > \frac{2}{I_1}\sqrt{mgr\, I_2 \cos\theta_0}$$

SO WE NEED $\omega_1 > \dfrac{2}{I_1}\sqrt{mgr\, I_2}$ FOR 'SLEEPING'

IF $\omega_1 < \omega_{1\,MIN}$, THE TOP WILL JUST FALL DOWN.

WE CAN VERIFY THE CONDITION FOR $\omega_{1\,MIN}$ DIRECTLY BY CONSIDERING V_{EFF} IN THE SMALL ANGLE APPROXIMATION.

WE SUPPOSE $L_z \sim L_1 = I_1\omega_1$

$$V_{EFF} = \frac{(L_z - L_1\cos\theta)^2}{2 I_2 \sin^2\theta} + mgr\cos\theta \sim \frac{I_1^2\omega_1^2}{2I_2}\frac{\theta^2}{4} + mgr\left(1 - \frac{\theta^2}{2}\right)$$

$$\sim mgr + \frac{\theta^2}{2}\left(\frac{I_1^2\omega_1^2}{4I_2} - mgr\right)$$

FOR STABILITY WE NEED V_{EFF} PARABOLIC UPWARDS

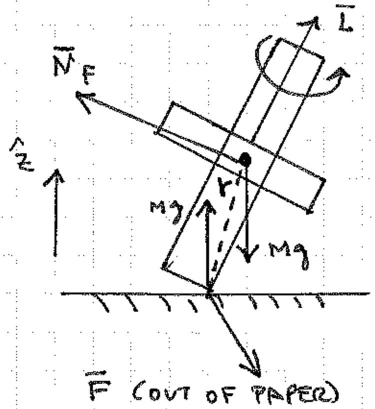
$$\Rightarrow \frac{I_1^2\omega_1^2}{4I_2} > mgr \quad \text{OR} \quad \omega_1 > \frac{2}{I_1}\sqrt{mgr\, I_2} \quad \text{AS BEFORE.}$$

4) RISING

IN THE NEXT TWO TOPICS WE CONSIDER TOPS WHICH ARE FREE TO ROLL ON A HORIZONTAL SURFACE.

WE ARE NOT PRIMARILY INTERESTED IN THE CASE OF ROLLING WITHOUT SLIPPING (SET 9, PROB 5) WHICH LEADS TO CIRCULAR MOTION OF THE C.M. INSTEAD WE CONSIDER THE SURPRISING EFFECTS WHEN SLIPPING OCCURS.

WE ARE NOT INTERESTED IN THE HORIZONTAL MOTION OF THE C.M., SO IMAGINE A TOP WHICH IS SLIPPING WITHOUT ROLLING AT THE POINT OF CONTACT. (USE A HORIZONTALLY ACCELERATED FRAME??)



WE SUPPOSE THE AXIS OF THE TOP HAS A BLUNT END AS SHOWN. FRICTION OPPOSES THE VELOCITY OF CONTACT, AND IS OUT OF THE PAPER. THE TORQUE DUE TO THIS FRICTION (ABOUT THE C.M.) IS $\bar{N}_F = \bar{r} \times \bar{F}$ WHICH IS IN THE PLANE OF $\hat{z}$ AND $\bar{L}$.

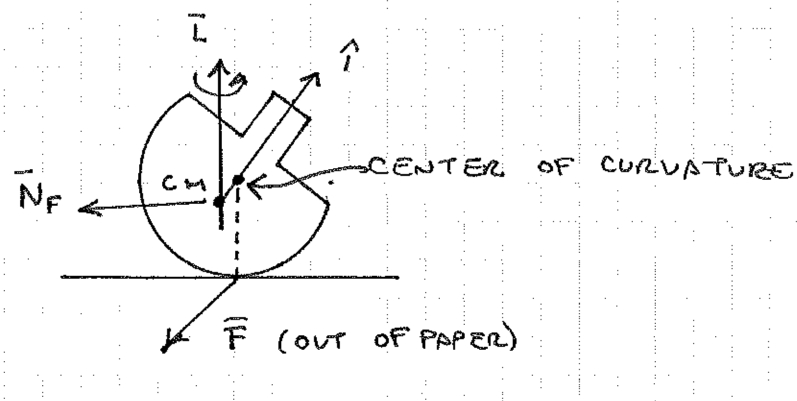
HENCE IN ADDITION TO THE PRECESSION OF $\bar{L}$ ABOUT THE VERTICAL DUE TO THE TORQUE OF $\bar{r} \times m\bar{g}$, THE TOP "PRECESSES" ABOUT A HORIZONTAL AXIS, CAUSING $\bar{L}$ (AND HENCE $\hat{1}$ FOR A FAST TOP) TO RISE TO THE VERTICAL.

THE SLIPPING FRICTION EATS UP THE ROTATIONAL KINETIC ENERGY — SOME OF WHICH IS ALSO TRANSFERRED TO POTENTIAL ENERGY AS THE TOP RISES.

5) THE TIPPIE TOP [B & O SEC 6-13]

IN THE RISING TOP THE C.M. IS ALWAYS ABOVE THE CENTER OF ROTATION OF THE RISING MOTION — WHICH IS JUST THE POINT OF CONTACT WITH THE FLOOR.

IN THE TIPPIE TOP THE C.M. IS INITIALLY BELOW THE CENTER OF ROTATION. THE FORCE OF CONTACT STILL CAUSES THE C.M. TO RISE — WHICH MAKES THE TOP FLIP OVER!



WE IGNORE THE SLOW PRECESSION OF THE TOP COMPARED TO THE EFFECT OF SLIDING FRICTION.

WE IMAGINE THE TIPPIE TOP IS SET SPINNING SO THAT THE ANGULAR MOMENTUM IS NEARLY VERTICAL.

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The tippie top both slides and rolls – and the C.M. moves in a horizontal circle. This causes the velocity of contact to rotate in a horizontal plane, and hence the torque due to sliding friction also rotates about the vertical i.e. $\langle \bar{N}_F \rangle = 0$

Since $\frac{d\bar{L}}{dt} = \bar{N}$, we get $\left\langle \frac{d\bar{L}}{dt} \right\rangle \approx 0 \Rightarrow \bar{L} \sim$ constant.

So $\bar{L}$ remains essentially vertical at all times.

This is clearly different from the rising top where $\bar{L}$ rose to the vertical.

In the tippie top, the <u>body</u> <u>axes</u> rotate even though $\bar{L} \sim$ constant, until the symmetry axis $\hat{1}$ has been flipped by 180°. This sounds a bit like free precession — but in free precession $\hat{1}$ precesses about $\bar{L}$, not about a perpendicular to $\bar{L}$!

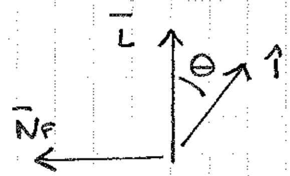
Let's go back to Euler's equations:

$$\bar{N} = \frac{\delta \bar{L}}{\delta t} + \bar{\omega} \times \bar{L}.$$

We suppose the top is almost spherical: $I_1 \sim I_2 = I_3$

So $\bar{L} \sim I \bar{\omega}$. Then $\bar{L} \sim$ const $\Rightarrow \bar{\omega} \sim$ const, and $\bar{\omega}$ always remains vertical also.

$$\therefore \quad \bar{N} \sim \left.\frac{\delta \bar{L}}{\delta t}\right|_{\text{body frame}}$$



In the <u>body frame</u>, $\bar{L}$ is not constant! The frictional torque $\bar{N}_F$ pulls $\bar{L}$ away from $\hat{1} \Rightarrow \theta$ <u>increases</u>. When θ has reached 180° the top has flipped!

Using this analysis, B & O estimate the time to flip ~4 sec.

We demonstrate the tippie top effect with a hard-boiled egg.

[See also, Amateur Scientist, Sci. Am Oct '79 & March '81]

A detailed discussion is given by R.J. Cohen, Am. J. Phys <u>45</u>, 12 (1977).