Ph 205 Lecture 18

Motion of a Rigid Body with no External Torques

In our example of the rotating earth, we found a precession of the axis of rotation even if there were no external torques. Another familiar example is the flight of a football: only a well-thrown football spirals nicely — a small error leads to a wobbly pass.

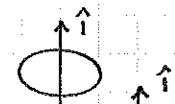
Of course, if the torque $\bar{N} = 0$, then $\bar{L}$ = constant, so something is invariant. The difficulty is that $\bar{L} = \bar{\bar{I}} \cdot \bar{\omega}$, so $\bar{L}$ is not $\parallel$ to $\bar{\omega}$ unless $\bar{\omega}$ is along a principal axes.

i.e., even if $\bar{N}=0$, $\bar{\omega}$ is constant only for rotations about a principal axis.

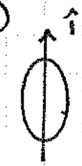
Symmetric Top [B & O sec. 6.9 ; L & L sec. 33]

We continue our consideration of a body with a symmetry axis, which we call axis $\hat{1}$. Then $I_1 \neq I_2$, but $I_2 = I_3$ for the principal moments of inertia.

$I_1 > I_2$ for the earth, a coin, or any oblate object



$I_1 < I_2$ for a football, a pencil, or any prolate object



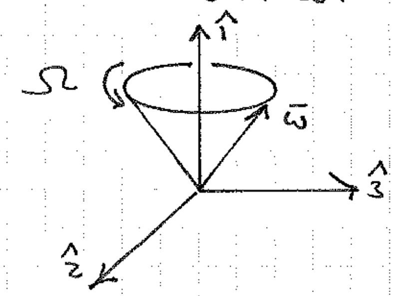
Measuring components of angular velocity $\bar{\omega}$ along the body frame axes, we use Euler's equations (p. 187) to find

$$\left.\begin{aligned} \dot{\omega}_1 &= 0 \\ \dot{\omega}_2 &= -\omega_1 \omega_3 \frac{(I_1 - I_2)}{I_2} \\ \dot{\omega}_3 &= +\omega_1 \omega_2 \frac{(I_1 - I_2)}{I_2} \end{aligned}\right\} \text{using } I_2 = I_3$$

or $$\dot{\bar{\omega}} = \frac{I_1 - I_2}{I_2} \omega_1 \hat{1} \times \bar{\omega} \equiv \bar{\Omega} \times \bar{\omega}$$

where $$\bar{\Omega} = \frac{I_1 - I_2}{I_2} \omega_1 \hat{1}$$

Thus in the body frame, $\bar{\omega}$ precess about the symmetry axis $\hat{1}$ (even if $\bar{\omega}$ originally pointed along $\hat{2}$....)



$\leftarrow I_1 > I_2$

WE NOW ASK WHAT AN INERTIAL OBSERVER WOULD SEE?

THE INERTIAL OBSERVER HAS A HARD TIME IDENTIFYING THE $\bar{\omega}$ OF A TUMBLING OBJECT, BUT (S)HE CAN MORE EASILY SPOT THE DIRECTIONS OF THE BODY AXES – AS THEY ARE FIXED TO THE BODY.

SO WHERE DO $\hat{1}, \hat{2}$ AND $\hat{3}$ (AS WELL AS $\bar{\omega}$) POINT?

IN THE <u>INERTIAL</u> FRAME, $\bar{L} = \bar{\bar{I}} \cdot \bar{\omega}$ = CONSTANT VECTOR $(\bar{N} = 0)$

AND $\bar{L} = I_1 \omega_1 \hat{1} + I_2 (\omega_2 \hat{2} + \omega_3 \hat{3})$

WHILE $\bar{\omega} = \omega_1 \hat{1} + \omega_2 \hat{2} + \omega_3 \hat{3}$

SO $\bar{L} = I_1 \omega_1 \hat{1} + I_2 (\bar{\omega} - \omega_1 \hat{1}) = (I_1 - I_2)\omega_1 \hat{1} + I_2 \bar{\omega}$

$\Rightarrow$ <u>$\bar{\omega}$, $\bar{L}$ AND $\hat{1}$ ALL LIE IN A PLANE</u>

NOTE $\bar{\omega} = \dfrac{\bar{L}}{I_2} - \dfrac{I_1 - I_2}{I_2} \omega_1 \hat{1} = \dfrac{\bar{L}}{I_2} - \bar{\Omega}$

HOW DO $\bar{\omega}$ AND $\hat{1}$ MOVE RELATIVE TO THE CONSTANT $\bar{L}$?

RECALL THAT $\dot{\bar{\omega}}\big|_{\text{INERTIAL FRAME}} = \dot{\bar{\omega}}\big|_{\text{BODY FRAME}}$

SO $\dot{\bar{\omega}} = \bar{\Omega} \times \bar{\omega}$ HOLDS IN THE INERTIAL FRAME AS WELL.

BUT $\bar{\Omega} = \dfrac{\bar{L}}{I_2} - \bar{\omega} \implies \underline{\dot{\bar{\omega}} = \dfrac{\bar{L}}{I_2} \times \bar{\omega}}$ $[\bar{\omega} \times \bar{\omega} = 0]$

THAT IS, $\bar{\omega}$ PRECESSES ABOUT $\bar{L}$ WITH ANGULAR VELOCITY

$\bar{\omega}_p = \bar{L}/I_2$

SIMILARLY $\dot{\bar{\Omega}} = -\dot{\bar{\omega}} = -\dfrac{\bar{L}}{I_2} \times \bar{\omega} = \dfrac{\bar{L}}{I_2} \times \bar{\Omega}$ SINCE $\dot{\bar{L}} = 0$

BUT $\bar{\Omega} = \Omega \hat{1}$ WHERE Ω = CONSTANT

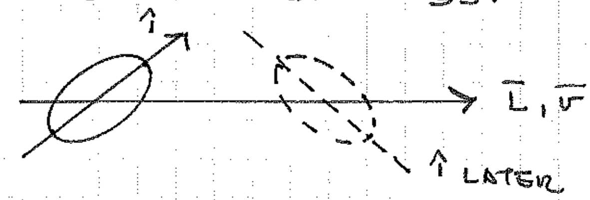
$\implies \underline{\dot{\hat{1}} = \bar{L}/I_2 \times \hat{1}}$

THUS PRINCIPAL AXIS $\hat{1}$ ALSO PRECESSES ABOUT $\bar{L}$ WITH ANGULAR VELOCITY $\bar{\omega}_p$. THIS KEEPS $\bar{L}$, $\bar{\omega}$ AND $\hat{1}$ ALWAYS IN A PLANE.

<u>EXERCISE</u>: SHOW THAT IN THE BODY FRAME, ANGULAR MOMENTUM $\bar{L}$ APPEARS TO PRECESS ABOUT THE $\hat{1}$ AXIS ACCORDING TO $\dfrac{\delta \bar{L}}{\delta t} = \bar{\Omega} \times \bar{L}$

Ph 205 LECTURE 18

THIS EXPLAINS A WOBBLY FOOTBALL PASS:



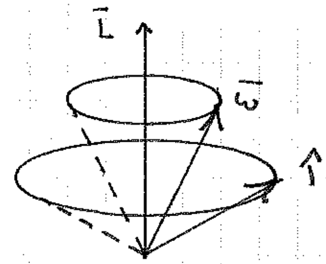
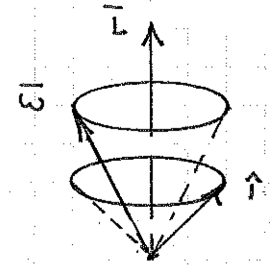
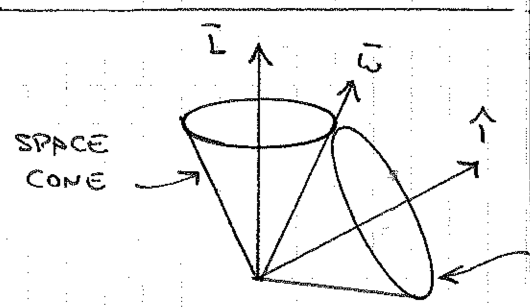
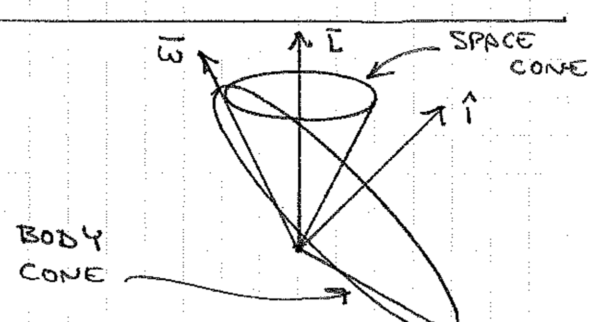
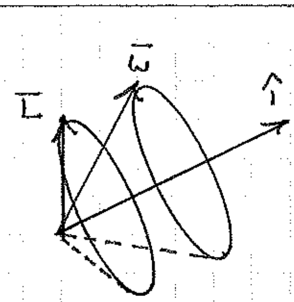
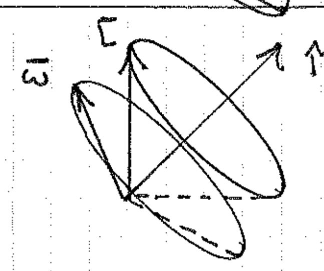
PEOPLE WITH TWISTED MINDS HAVE CONSTRUCTED ANOTHER WAY TO VISUALIZE WHAT'S HAPPENING. THIS ALSO ALLOWS US TO KEEP TRACK OF THE ORIENTATION OF AXES $\hat{2}$ AND $\hat{3}$. THE INGREDIENTS ARE:

a) $\bar{\omega}$ PRECESSES ABOUT $\bar{L}$, ACCORDING TO AN INERTIAL OBSERVER. THE CONICAL PATH OF $\bar{\omega}$ IS CALLED THE <u>SPACE CONE</u>.

b) $\bar{\omega}$ PRECESSES ABOUT $\hat{1}$ ACCORDING TO AN OBSERVER IN THE BODY FRAME. CALL THIS CONICAL PATH THE <u>BODY CONE</u>

c) WE KNOW THAT $\bar{L}$, $\bar{\omega}$ AND $\hat{1}$ ALWAYS LIE IN A PLANE. SO WE CAN MAKE ALL THIS CONSISTENT IF <u>THE BODY CONE ROLLS WITHOUT SLIPPING ON THE SPACE CONE</u>.

FOR A <u>PROLATE</u> OBJECT, $\bar{\omega}$ & $\hat{1}$ ARE ON THE SAME SIDE OF $\bar{L}$.
FOR AN <u>OBLATE</u> OBJECT, $\bar{\omega}$ & $\hat{1}$ ARE ON OPPOSITE SIDES OF $\bar{L}$.

	PROLATE	OBLATE
INERTIAL OBSERVER $\bar{L}$ = CONST.		
SPACE & BODY CONE VIEWPOINT		
BODY FRAME OBSERVER $\hat{1}$ = CONST.		

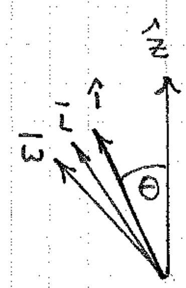
TO MAKE $\bar{L}$ PRECESS ABOUT $\hat{1}$, THERE IS A 'FICTITIOUS' TORQUE.

EXAMPLE: THE 'DIAMETER' OF THE NORTH POLE

WE MAY APPLY THE IDEA OF SPACE AND BODY CONES TO THE STEADY PRECESSION OF A SYMMETRIC TOP DUE TO AN EXTERNAL TORQUE.

IN THIS CASE $\bar{L}$ IS NOT CONSTANT, BUT RATHER PRECESSES ABOUT THE "VERTICAL". THE CONSTANT 'VERTICAL' DIRECTION $\hat{z}$ WILL NOW PLAY THE ROLE OF THE AXIS OF THE SPACE CONE.

DUE TO THE PRECESSION, THE TOTAL ANGULAR VELOCITY IS



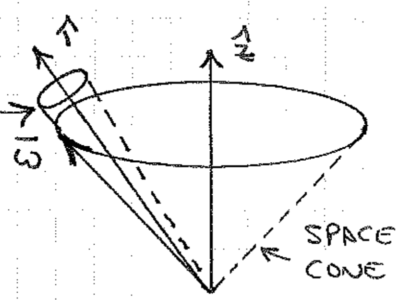
$$\bar{\omega} = \omega_0 \hat{1} + \Omega \hat{z} \iff \text{ALL IN A PLANE}$$

WHERE ω_0 = ANGULAR VELOCITY ABOUT AXIS 1 IF NO TORQUE

Ω = PRECESSION ANGULAR VELOCITY.

[CAN YOU SHOW THAT $\bar{L} = \bar{\bar{I}} \cdot \bar{\omega} = (I_1 \omega_0 + (I_1 - I_2)\Omega \cos\theta)\hat{1} + I_2 \Omega \hat{z}$
SO THAT $\hat{1}, \hat{z}, \bar{L}$ AND $\bar{\omega}$ ARE ALL IN A PLANE]

FOR THE CASE OF AN OBLATE TOP WHERE Ω IS OPPOSITE TO THE SENSE TO ω_0, THE VECTORS ARE AS SHOWN.



WE CALL THE MOTION OF $\bar{\omega}$ ABOUT THE CONSTANT VECTOR $\hat{z}$ THE SPACE CONE.

AGAIN THE MOTION OF $\bar{\omega}$ ABOUT $\hat{1}$ IN THE BODY FRAME IS CALLED THE BODY CONE.

CONSIDER THE PRECESSION OF THE EQUINOXES OF THE EARTH. (p181). THE PRECESSION Ω IS INDEED OPPOSITE TO THE SENSE OF ROTATION ω_0. (CAN YOU SHOW THIS?) THE ANGLE θ IS $23\frac{1}{2}°$.

BY 'DIAMETER OF THE NORTH POLE', WE MEAN THE DIAMETER OF THE BODY CONE AT THE SURFACE OF THE EARTH.

IN ONE REVOLUTION ABOUT THE SPACE CONE (26000 YEARS) THE BODY CONE REVOLVES 365 x 26000 TIMES. HENCE

$$d_{\text{POLE}} = \frac{\text{DIAMETER OF SPACE CONE}}{(365)(26000)} = \frac{2\,(6000\,\text{KM})\sin(23\frac{1}{2}°)}{(365)(26000)} = \frac{1}{2}\ \text{METER}$$

[WE HAVE IGNORED THE FACT THAT $\bar{\omega}$ AND $\hat{1}$ ARE ABOUT 10 METERS APART DUE TO CAUSES UNRELATED TO THE PRECESSION OF THE EQUINOXES.]

ASYMMETRIC TOP (L&L SEC. 37)

WE RETURN TO MOTION WITH NO EXTERNAL TORQUES, $\bar{N}=0$, AND CONSIDER AN OBJECT WITH PRINCIPAL MOMENTS OF INERTIA $I_1 \neq I_2 \neq I_3$ — LIKE A BOOK.

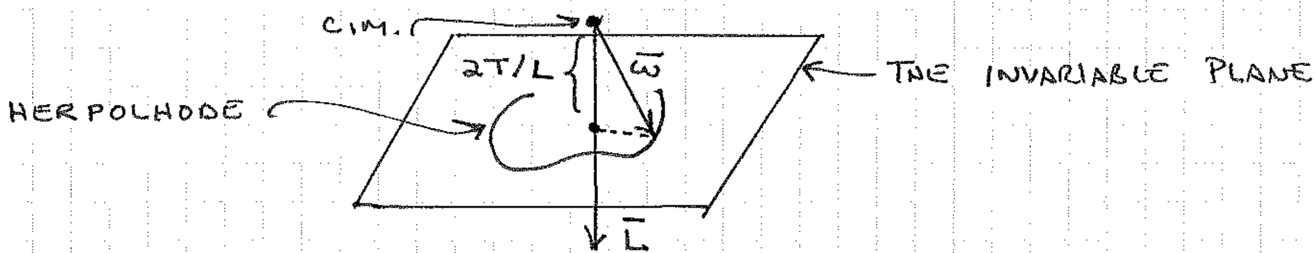
POINSOT (1834) HAS GIVEN A METHOD OF VISUALIZING THE MOTION.

RECALL THAT KINETIC ENERGY $T = \frac{1}{2}\bar{\omega}\cdot\bar{\bar{I}}\cdot\bar{\omega} = \frac{1}{2}\bar{L}\cdot\bar{\omega}$

THEN IF $\bar{N}=0$, KINETIC ENERGY IS CONSERVED, AS WELL AS $\bar{L}$.

$$\left[T = \sum_i \tfrac{1}{2} m_i \bar{v}_i^2 \Rightarrow \frac{dT}{dt} = \sum_i m_i \bar{v}_i\cdot\dot{\bar{v}}_i = \sum_i m_i(\bar{\omega}\times\bar{r}_i)\cdot\dot{\bar{v}}_i = \bar{\omega}\cdot\sum_i \bar{r}_i\times m_i\dot{\bar{v}}_i = \bar{\omega}\cdot\bar{N}\right]$$

A GEOMETRIC INTERPRETATION IS THAT $\bar{\omega}$ IS CONSTRAINED TO MOVE IN A PLANE $\perp$ TO THE CONSTANT VECTOR $\bar{L}$.



FOR A SYMMETRIC BODY WE KNOW THAT THE PATH OF $\bar{\omega}$ IN THE PLANE IS A CIRCLE. POINSOT CALLED THE PATH IN THE GENERAL CASE THE <u>HERPOLHODE</u> (= SNAKE PATH).

THE PLANE IS CALLED THE <u>INVARIABLE PLANE</u>.

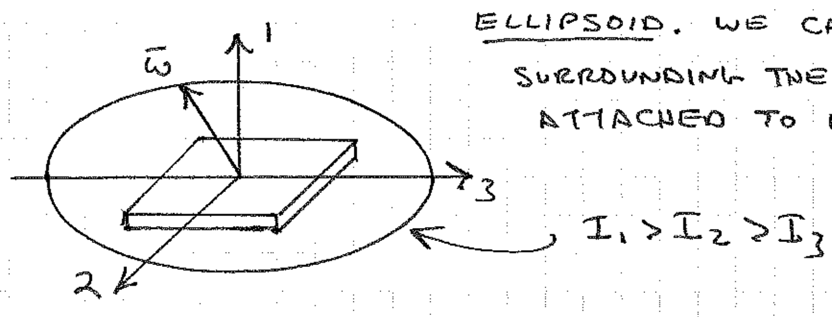
IF WE USE THE PRINCIPAL AXES AS COORDINATE AXES, THEN

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

THUS IN THE <u>BODY-COORD. SYSTEM</u>, $\bar{\omega}$ IS RESTRICTED TO MOVE OVER THE SURFACE OF AN ELLIPSOID:

$$\frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = 1$$

THIS IS SOMETIMES CALLED THE <u>INERTIA ELLIPSOID</u>, OR THE <u>ENERGY ELLIPSOID</u>. WE CAN IMAGINE IT AS SURROUNDING THE BODY AND RIGIDLY ATTACHED TO IT.



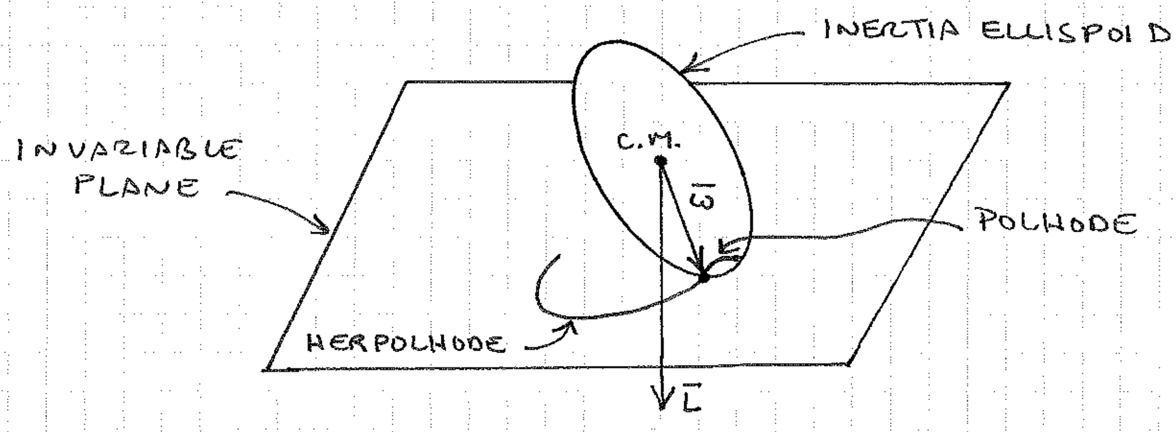
A MATHEMATICAL MIRACLE IS THAT THE NORMAL TO THE ELLIPSOID IS PARALLEL TO $\bar{L}$!

WE HAVE $f(\omega_1, \omega_2, \omega_3) = 1$ WHERE $f = \dfrac{\omega_1^2}{2T/I_1} + \dfrac{\omega_2^2}{2T/I_2} + \dfrac{\omega_3^2}{2T/I_3}$

THE NORMAL IS PARALLEL TO $\bar{\nabla} f = \left(\dfrac{\partial f}{\partial \omega_1}, \dfrac{\partial f}{\partial \omega_2}, \dfrac{\partial f}{\partial \omega_3} \right)$

$$= \frac{1}{T} \left(I_1\omega_1, I_2\omega_2, I_3\omega_3 \right) = \frac{\bar{L}}{T}$$

HENCE THE INERTIA ELLIPSOID IS TANGENT TO THE INVARIABLE PLANE.



ALL THAT CAN HAPPEN IS THAT THE INERTIA ELLIPSOID ROLLS AROUND WITHOUT SLIPPING — BUT ITS CENTER STAYS AT A FIXED POINT ABOVE THE INVARIABLE PLANE.

THE POINT OF TANGENCY WITH THE INVARIABLE PLANE SWEEPS OUT A CLOSED CURVE ON THE INERTIA ELLIPSOID, CALLED THE <u>POLHODE</u>.

AS THEY SAY: "THE <u>POLHODE</u> OF THE <u>INERTIA</u> <u>ELLIPSOID</u> ROLLS WITHOUT SLIPPING ON THE <u>HERPOLHODE</u> LYING IN THE <u>INVARIABLE</u> <u>PLANE</u>." GOT IT?

SINCE THE PRINCIPAL AXES ARE FIXED WITH RESPECT TO THE INERTIA ELLIPSOID, WE CAN UNDERSTAND THE RELATION BETWEEN $\bar{L}$, $\bar{\omega}$ AND $\hat{1}, \hat{2}, \hat{3}$ AS THE BODY ROTATES. POINSOT'S METHOD DOES NOT TELL US <u>WHEN</u> A PARTICULAR METHOD HOLDS, AS IT IS BASED ON CONSERVATION LAWS. IN PRINCIPLE WE COULD INTEGRATE EULER'S EQUATIONS TO FOLLOW THE MOTION IN TIME ALONG THE POLHODE OR HERPOLHODE. (SEE ALSO ROUTH, "ADVANCED RIGID DYNAMICS", SECS 196-198)

THE FORM OF THE POLHODES

IT IS NOT TOO HARD TO GET A ROUGH PICTURE OF THE FORM OF THE POLHODES ON THE INERTIA ELLIPSOID.

THE POLHODES ARE FUNCTIONS OF THE KINETIC ENERGY OF ROTATION T, AND THE ANGULAR MOMENTUM $\vec{L}$.

THE ENERGY ELLIPSOID IS $2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$

THE CONSTANT L^2 CAN ALSO BE EXPRESSED AS AN ELLIPSOID:

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

THE POLHODES ARE THE INTERSECTIONS OF THESE TWO ELLIPSOIDS!

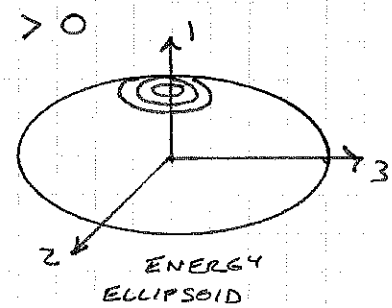
IN CASE $\omega_i \gg \omega_j, \omega_k$, WE CAN FIND THE POLHODE NEAR AXIS i ON ELIMINATING ω_i ABOVE.

WE CONSIDER THE CASE WHERE $I_1 > I_2 > I_3$

a) ELIMINATE ω_1

$$2TI_1 - L^2 = I_2(I_1 - I_2)\omega_2^2 + I_3(I_1 - I_3)\omega_3^2 > 0$$

WHEN ω_1 IS BIG, ω_2, ω_3 ARE SMALL AND THE POLHODES ARE ELLIPSES ABOUT $\hat{1}$



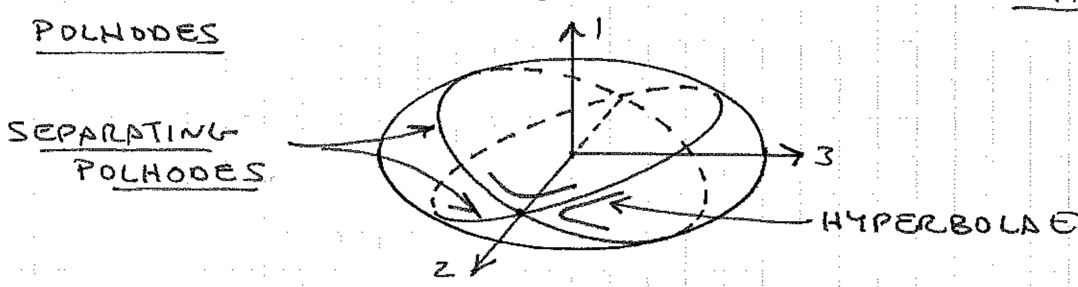
b) ELIMINATE ω_2

$$2TI_2 - L^2 = I_1\underbrace{(I_2 - I_1)}_{<0}\omega_1^2 + I_3\underbrace{(I_2 - I_3)}_{>0}\omega_3^2$$

IF ω_2 IS BIG, ω_1, ω_3 ARE SMALL AND POLHODES $\Leftrightarrow$ HYPERBOLAE

A SPECIAL CASE IS $2TI_2 - L^2 = 0 \Rightarrow \omega_1 = \pm\sqrt{\dfrac{I_3(I_2 - I_3)}{I_1(I_1 - I_3)}}\,\omega_3$

THE CORRESPONDING POLHODES ARE CALLED THE SEPARATING POLHODES

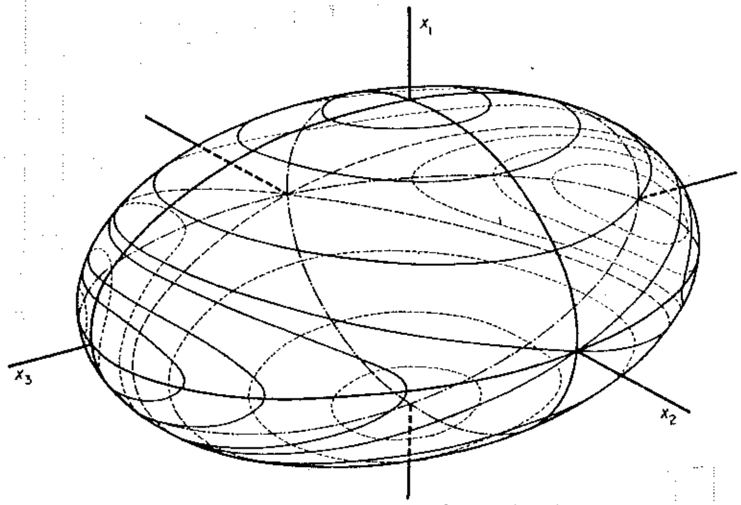


THEY ARE THE INTERSECTIONS OF TWO CROSSED PLANES WITH THE ENERGY ELLIPSOID.

c) ELIMATE ω_3

$$L^2 - 2T I_3 = I_1(I_1 - I_3)\omega_1^2 + I_2(I_2 - I_3)\omega_2^2 > 0$$

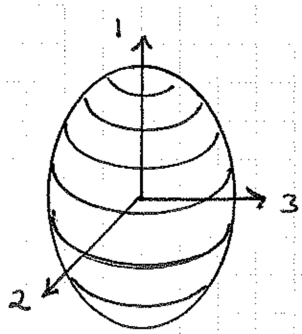
SO IF ω_3 IS BIG, ω_1 & ω_2 ARE SMALL $\Rightarrow$ POLHODES ARE ELLIPSES.



IF THE BODY IS SYMMETRIC ABOUT AN AXIS, THE POLHODES ARE, OF COURSE, CIRCLES

I.e. IF $I_2 = I_3$ THEN a) $\Rightarrow$ $\omega_2^2 + \omega_3^2 =$ CONSTANT

b) $\Rightarrow$ $\omega_1 =$ CONSTANT



SO POINSOT'S METHOD REDUCES TO OUR PREVIOUS APPROACH IN THIS CASE.

STABILITY OF ROTATIONS ABOUT A PRINCIPAL AXIS (B&O SEC 6.7)

OUR PICTURE OF THE POLHODES OF AN ASYMMETRIC TOP SUGGESTS THAT ROTATIONS ABOUT THE PRINCIPAL AXES OF GREATEST AND LEAST MOMENTS (I_1 & I_3) ARE STABLE — WHILE ROTATION ABOUT THE INTERMEDIATE PRINCIPAL AXIS IS UNSTABLE. TRY IT WITH A BOOK, A PLAYING CARD, A TENNIS RACKET...

LET'S TRY TO VERIFY THIS USING EULER'S EQUATIONS.

IN THIS ARGUMENT WE DO NOT SUPPOSE THE PRINCIPAL AXES ARE CHOSEN IN ANY PARTICULAR ORDER.

WE CONSIDER ROTATIONS WHERE ω_1 IS BIG INITIALLY, SO ω_2 & ω_3 ARE SMALL, AND $\omega_2\omega_3$ IS NEGLIBLE.

EULER'S EQ. $\Rightarrow$ $I_1\dot{\omega}_1 \sim 0 \Rightarrow \omega_1 \sim$ CONSTANT

$$I_2\dot{\omega}_2 = -\omega_1\omega_3(I_1 - I_3)$$

$$I_3\dot{\omega}_3 = -\omega_1\omega_2(I_2 - I_1)$$

FOR SMALL ω_2, ω_3 WE HOPE FOR OSCILLATORY SOLUTIONS (STABILITY)

$$\omega_2 = A_2 e^{i\Omega t} \qquad \omega_3 = A_3 e^{i\Omega t} \qquad (A_2, A_3 \text{ COMPLEX})$$

EULER $\Rightarrow$ $$i\Omega A_2 = -\omega_1\frac{(I_1 - I_3)}{I_2}A_3$$

$$i\Omega A_3 = -\omega_1\frac{(I_2 - I_3)}{I_3}A_2$$

$$i\Omega A_2 = \omega_1^2\left(\frac{I_1 - I_3}{I_2}\right)\left(\frac{I_2 - I_1}{I_3}\right)\frac{A_2}{i\Omega}$$

$$\Omega^2 = -\omega_1^2\left(\frac{I_1 - I_3}{I_2}\right)\left(\frac{I_2 - I_1}{I_3}\right)$$

THE MOTION IS OSCILLATORY ONLY IF $\Omega^2 > 0$

[IF $\Omega^2 < 0$, $\omega_2 \sim A_2 e^{\lambda t}$ WITH λ REAL

THEN ω_2, ω_3 GROW LARGE $\Rightarrow$ UNSTABLE MOTION.

EVENTUALLY $\omega_2\omega_3$ CANNOT BE IGNORED AND OUR APPROXIMATION FAILS.]

THE STABLE CASES ARE

a) $I_1 > I_2 > I_3$ } ROTATION ABOUT THE AXIS OF GREATEST OR LEAST MOMENT (REMEMBER $\bar{\omega} \sim \omega_1 \hat{1}$)

b) $I_1 < I_2 < I_3$

THE UNSTABLE CASES ARE

a) $I_2 > I_1 > I_3$ } ROTATION ABOUT AN INTERMEDIATE AXIS

b) $I_3 > I_1 > I_2$

SPECIAL CASES

a) $I_1 = I_2 = I_3$ THEN $\dot{\bar{\omega}} = 0 \Rightarrow \bar{\omega} = $ CONSTANT AND THE ROTATION IS STABLE

OF COURSE, THE OBJECT IS A SPHERE, SO IT'S NOT SURPRISING.

b) $I_1 = I_2$ (OR $I_1 = I_3$)

IF $I_1 = I_2$ THEN $\dot{\omega}_3 = 0 \Rightarrow \dot{\omega}_2 = $ CONSTANT

$\Rightarrow$ ω_2 GROWS BIG $\Rightarrow$ CANNOT IGNORE $\omega_2 \omega_3$

$\Rightarrow \dot{\omega}_1 \neq 0$

OF COURSE THE NET RESULT IS THAT $\bar{\omega}$ PRECESSES ABOUT AXIS 3, WHICH IS THE SYMMETRY AXIS.

THIS MOTION IS UNSTABLE IN THE SENSE THAT ω_2 DOES NOT REMAIN SMALL - EVEN THO $\omega_3 = $ CONST.