

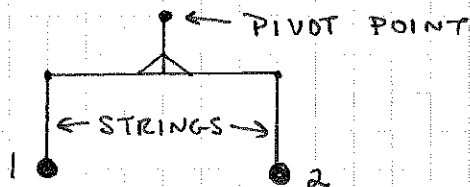
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COUPLED OSCILLATIONS

REFERENCES: B&O sec 2-11 ; L&L sec 23,24

THE INVESTIGATION OF COUPLED OSCILLATIONS SEEMS TO HAVE BEGUN WITH DANIEL BERNOULLI (1753) WITH THE PARTICULAR EXAMPLE OF A BEAM BALANCE WITH 2 HANGING WEIGHTS.



HE NOTED THAT WHEN MASS 1 IS DISPLACED & LET SWING, AT FIRST IT MOVES LIKE A SIMPLE PENDULUM, BUT AFTER A WHILE THIS MOTION DIES OUT — AND MASS 2 BEGINS TO SWING LIKE A SIMPLE PENDULUM. AFTER A WHILE THE OSCILLATION IS 'TRANSFERRED' BACK TO MASS 1, AND SO ON (IF DAMPING IS SMALL).

WHILE THIS 'TRANSFERRENCE OF OSCILLATION' IS A VERY DRAMATIC EFFECT, BERNOULLI REALISED THAT ANOTHER FEATURE OF THE MOTION IS MORE FUNDAMENTAL. IT IS POSSIBLE TO SET BOTH MASSES GOING IN SUCH A MANNER THAT THE WHOLE SYSTEM (INCLUDING THE BALANCE BEAM) OSCILLATES WITH THE SAME FREQUENCY. FURTHERMORE, ONCE THIS MOTION IS ESTABLISHED IT WILL CONTINUE WITHOUT ANY TRANSFERRENCE OF OSCILLATION.

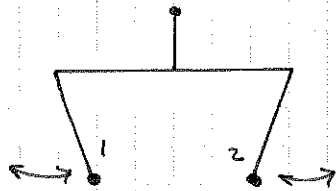
IN FACT THERE ARE 3 SUCH SPECIAL FREQUENCIES FOR WHICH THE WHOLE SYSTEM VIBRATES SYNCHRONOUSLY. THIS IS AN ILLUSTRATION OF A GENERAL PRINCIPAL:

A SYSTEM OF COUPLED OSCILLATORS OF n DEGREES OF FREEDOM (OF OSCILLATORY MOTION) HAS n CHARACTERISTIC MOTIONS IN WHICH ALL PARTS OF THE SYSTEM OSCILLATE WITH THE SAME FREQUENCY, ω_i , $i=1 \dots n$. THESE n CHARACTERISTIC MOTIONS ARE CALLED NORMAL MODES. THE ω_i ARE CALLED THE NORMAL FREQUENCIES OR EIGEN FREQUENCIES.

(THE AMPLITUDE OF OSCILLATION OF SOME PARTS OF THE SYSTEM MAY BE ZERO FOR SOME OF THE NORMAL MODES.)

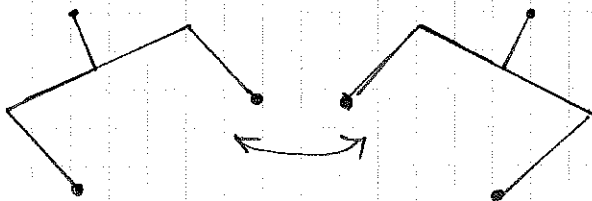
IN THE BEAM BALANCE EXAMPLE THERE ARE 3 DEGREES OF FREEDOM, AND HENCE 3 NORMAL MODES. IT IS EXTREMELY USEFUL TO GUESS THE NATURE OF THESE MODES BEFORE TRYING TO CALCULATE ANYTHING.

MODE 1



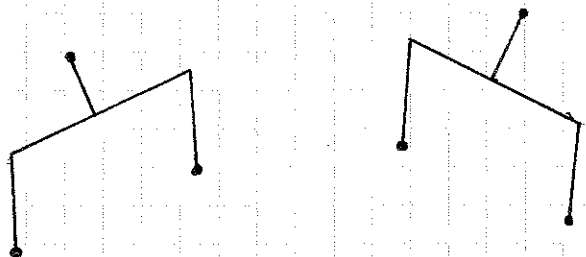
THE BEAM DOESN'T OSCILLATE,
MASSES 1 & 2 SWING
OPPOSITELY

MODE 2



ALL 3 MASSES
SWING TOGETHER

MODE 3



MASSES 1 & 2 SWING
TOGETHER, BUT OPPOSITE
TO THE SWING OF THE
BEAM

WE CAN ALSO GUESS THAT $\omega_1 > \omega_3 > \omega_2$. MODE 2 INVOLVES Ponderous MOTION OF THE WHOLE SYSTEM. MODE 1 INVOLVES QUICK MOTION OF ONE PART OF THE SYSTEM RELATIVE TO ANOTHER

DEMONSTRATION OF THE GENERAL PRINCIPLE

WE NOW SKETCH A PROOF THAT N DEGREES OF FREEDOM $\Rightarrow N$ NORMAL MODES. THE PROOF WILL SUGGEST ONE METHOD OF SOLUTION FOR THE NORMAL FREQUENCIES.

TO GIVE A GENERAL ARGUMENT WE RECALL LAGRANGE'S METHOD. WE USE q_i TO LABEL THE N INDEPENDENT COORDINATES. IF THE CONSTRAINTS OF THE SYSTEM ARE INDEPENDENT OF TIME, THEN THE KINETIC ENERGY CAN BE WRITTEN (SEE P 49)

$$T = \frac{1}{2} \sum_{i,j} a_{ij} \dot{q}_i \dot{q}_j \quad a_{ij} = a_{ji} = f(q) \text{ ONLY, NOT } f(q,t)$$

FOR SMALL OSCILLATIONS THE a_{ij} CAN BE REGARDED AS CONSTANTS. ANY DEVIATION OF THE a_{ij} FROM THEIR EQUILIBRIUM VALUES IS A HIGHER ORDER CORRECTION WHICH WE IGNORE. (UNTIL THE NEXT LECTURE).

FOR THE POTENTIAL ENERGY WE SUPPOSE THE q_i ARE CHOSEN SO $q_i = 0$ AT EQUILIBRIUM. WE THEN EXPAND TO 2ND ORDER:

$$V(q_i) = V(0) + \sum_i \frac{\partial V(0)}{\partial q_i} q_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V(0)}{\partial q_i \partial q_j} q_i q_j$$

AT EQUILIBRIUM, $\frac{\partial V(0)}{\partial q_i} = 0$. WE DEFINE $K_{ij} = \frac{\partial^2 V(0)}{\partial q_i \partial q_j} = K_{ji}$

$$\text{THEN } L = T - V = \frac{1}{2} \sum_{i,j} a_{ij} \dot{q}_i \dot{q}_j - V(0) - \frac{1}{2} \sum_{i,j} K_{ij} q_i q_j$$

THE EQUATIONS OF MOTION ARE $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

$$\text{OR } \underline{\sum_j a_{ij} \ddot{q}_j + \sum_j K_{ij} q_j = 0} \quad (\text{THE FACTOR } \frac{1}{2} \text{ DISAPPEARS})$$

THESE ARE THE COUPLED OSCILLATOR EQUATIONS.

WE RECOGNIZE THE FORM BY THINKING OF CARTESIAN COORDINATES FOR WHICH $a_{ii} = m_i$ $a_{ij} = 0$ IF $i \neq j$

AND K_{ij} = FORCE CONSTANT OF SPRING CONNECTING i TO j
 K_{ii} = FORCE CONSTANT OF SPRING CONNECTING i TO A FIXED POINT

IF $K_{ij} = 0$ WHEN $i \neq j$ THERE WOULD BE NO COUPLING.

BERNOULLI SUGGESTS WE TRY A SOLUTION

$$q_j = q_{j0} \cos \omega t$$

WHERE ω IS THE SAME FOR ALL j . LET'S TRY IT.

$$\ddot{q}_j = -\omega^2 q_j$$

$$\text{SO WE HAVE } \sum_j \underbrace{(-\omega^2 a_{ij} + K_{ij})}_{n \times n \text{ MATRIX}} q_{j0} = 0$$

THIS IS A SET OF LINEAR EQUATIONS IN n UNKNOWN, q_{j0} . SINCE THE CONSTANTS ON THE RIGHT-HAND SIDE ARE ALL ZERO, WE WILL NOT HAVE A UNIQUE SOLUTION FOR THE q_{j0} . IF q_{j0} ARE SOLUTIONS, SO ARE $C q_{j0}$ WHERE C = CONSTANT. BUT THIS JUST MEANS THAT THE SOLUTIONS HAVE AMPLITUDES WHICH ARE INDEPENDENT OF THE FREQUENCY — AS EXPECTED FOR SMALL OSCILLATIONS!

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WE ALSO KNOW FROM MATRIX THEORY (LINEAR ALGEBRA) THAT FOR SOLUTIONS WITH $q_{j0} \neq 0$ TO EXIST, THE DETERMINANT OF THE $N \times N$ MATRIX MUST VANISH!

$$i.e. \quad | -\omega^2 a_{ij} + k_{ij} | = 0$$

OF COURSE, THE DETERMINANT WHEN EVALUATED IS AN N TH ORDER POLYNOMIAL IN ω^2 .

$$|DET| = A(\omega^2)^N + B(\omega^2)^{N-1} + \dots$$

THEREFORE THERE ARE N ROOTS - THE N NORMAL FREQUENCIES!

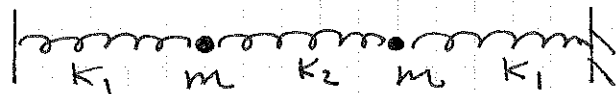
THE ROOTS ARE IN FACT REAL AND POSITIVE (L & L p67). IF THEY WEREN'T, ω WOULD HAVE AN IMAGINARY PART, LEADING TO SOLUTIONS LIKE $q_j \sim e^{\beta t}$ WHICH DIVERGE, & SO CLEARLY MUST NOT OCCUR. FORTUNATELY MATHEMATICS AGREES WITH THE REQUIREMENTS OF PHYSICAL REALITY IN THIS CASE.

IT CAN HAPPEN THAT A ROOT IS ZERO. THERE IS NO OSCILLATION THEN. THIS MEANS THAT ONE OF THE DEGREES OF FREEDOM DID NOT CORRESPOND TO AN OSCILLATORY MOTION, AND SHOULD HAVE BEEN LEFT OUT TO BEGIN WITH. (SEE PROBLEM 10, SET 7.)

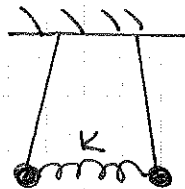
TO SUMMARIZE, A SYSTEM OF N DEGREES OF FREEDOM OF OSCILLATION HAS N NORMAL MODES. THE CORRESPONDING NORMAL FREQUENCIES CAN BE FOUND AS THE ROOTS OF AN N TH-ORDER POLYNOMIAL. FINALLY, THE RELATIVE AMPLITUDES OF THE MOTION OF THE COORDINATES IN ANY GIVEN MODE CAN BE FOUND AS THE SOLUTION TO A SET OF SIMULTANEOUS LINEAR EQUATIONS.



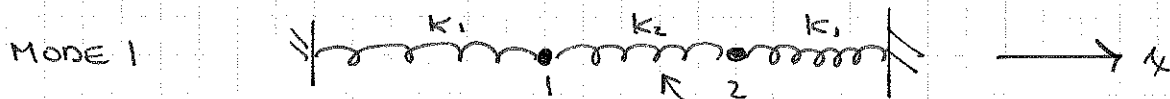
EXAMPLE 1. WE CONSIDER A FAMOUS CASE OF 2 DEGREES OF FREEDOM.



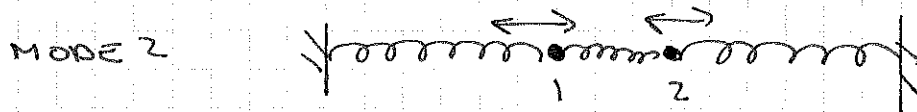
THIS IS EQUIVALENT TO THE COUPLED PENDULUM SYSTEM STUDIED IN PH 103 LAB



FIRST LET'S GUESS THE MODES:



THIS SPRING DOES NOT STRETCH } MASSES 1 & 2 MOVE TOGETHER AS IF RIGIDLY JOINED



MASSES 1 & 2 MOVE OPPOSITELY.

HAVING GUESSED THE MODES WE CAN AVOID THE METHOD WITH THE DETERMINANT!

MODE 1 LET x = DISPLACEMENT OF MASS 1 FROM EQUILIBRIUM.

THE RESTORING FORCE IS JUST $-k_1 x_1$, SINCE SPRING 2 DOES NOT CHANGE ITS LENGTH IN THIS MODE.

$$\therefore m \ddot{x}_1 = -k_1 x_1 \Rightarrow \omega_1 = \sqrt{\frac{k_1}{m}} \quad \text{ALSO } x_2 = x_1$$

MODE 2. NOW THE RESTORING FORCE ON MASS 1 IS $-k_1 x_1 - 2k x_1$, SINCE SPRING 2 IS COMPRESSED BY $2x_1$, WHEN MASS 1 MOVES BY x_1 ,

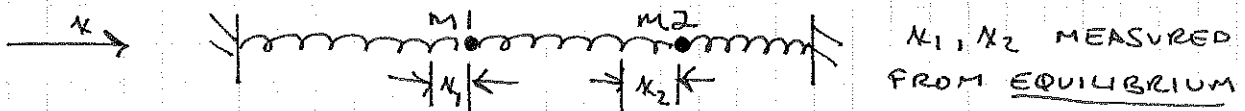
$$\therefore m \ddot{x}_1 = -(k_1 + 2k_2)x_1 \Rightarrow \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}} \quad \text{ALSO } x_2 = -x_1$$

LESSON: IF YOU CAN GUESS THE PRECISE FORM OF THE MODES, THE COUPLED OSCILLATOR PROBLEM CAN BE REDUCED TO A SERIES OF ONE DIMENSIONAL PROBLEMS.

EXAMPLE 2 NOW SUPPOSE $m_1 \neq m_2$ IN EXAMPLE 1.

WE STILL HAVE 2 MODES AND THEY ARE MORE OR LESS LIKE THOSE SKETCHED ABOVE - BUT SPRING 2 WILL BE COMPRESSED A BIT IN MODE 1, AND THE 2 MASSES WON'T HAVE EQUAL DISPLACEMENTS IN MODE 2.

SO WE MUST USE OUR GENERAL METHOD.



$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k_1 x_2 - k_2 (x_2 - x_1)$$

TRY $x_1 = A_1 \cos \omega t$ $x_2 = A_2 \cos \omega t$

$$-A_1 m_1 \omega^2 = -k_1 A_1 + k_2 A_2 - k_2 A_1$$

$$-A_2 m_2 \omega^2 = -k_1 A_2 - k_2 A_2 + k_2 A_1$$

$$\text{on } (K_1 + K_2 - m_1 \omega^2) A_1 - K_2 A_2 = 0$$

$$-K_2 A_1 + (K_1 + K_2 - m_2 \omega^2) A_2 = 0$$

FOR NON-TRIVIAL SOLUTIONS THE DETERMINANT MUST VANISH:

$$(K_1 + K_2 - m_1 \omega^2)(K_1 + K_2 - m_2 \omega^2) - K_2^2 = 0$$

$$m_1 m_2 \omega^4 - (K_1 + K_2)(m_1 + m_2) \omega^2 + K_1^2 + 2K_1 K_2 = 0$$

$$\text{AND } \omega^2 = \frac{(m_1 + m_2)(K_1 + K_2) \pm \sqrt{(K_1 + K_2)^2 (m_1 + m_2)^2 - 4m_1 m_2 (K_1^2 + 2K_1 K_2)}}{2m_1 m_2}$$

IF $m_1 = m_2$, $\omega^2 \rightarrow \frac{K_1 + K_2 \pm K_2}{m}$ AS BEFORE!

WE CAN NOW EXPLAIN THE 'TRANSFERENCE OF OSCILLATION'

SUPPOSE $m_1 = m_2 = m$. WE HAVE TWO SOLUTIONS, CORRESPONDING TO THE TWO MODES:

MODE 1 $x_1 = A_1 \cos \omega_1 t$ $x_2 = A_1 \cos \omega_1 t$

MODE 2 $x_1 = A_2 \cos \omega_2 t$ $x_2 = -A_2 \cos \omega_2 t$

THE GENERAL SOLUTION IS $x_1 = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$

$$x_2 = A_1 \cos \omega_1 t - A_2 \cos \omega_2 t$$

SUPPOSE AT $t=0$, $x_1 = A$, $x_2 = 0$, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

THEN $x_1 = \frac{A}{2} (\cos \omega_1 t + \cos \omega_2 t) = A \cos \frac{\omega_1 + \omega_2}{2} t \cos \frac{\omega_1 - \omega_2}{2} t$

$$x_2 = \frac{A}{2} (\cos \omega_1 t - \cos \omega_2 t) = A \sin \frac{\omega_1 + \omega_2}{2} t \sin \frac{\omega_2 - \omega_1}{2} t$$

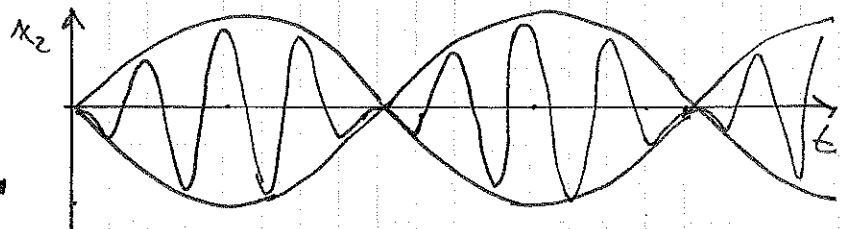
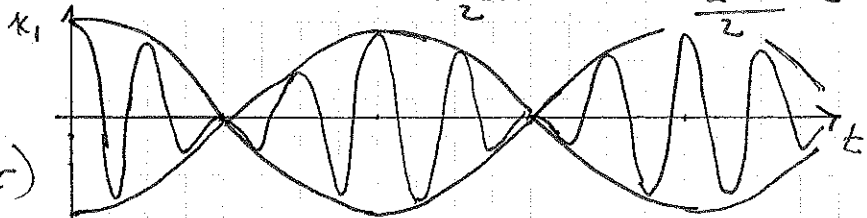
THE TRANSFERENCE IS DRAMATIC IF $\omega_1 \approx \omega_2$
 $(\Rightarrow K_2 \ll K_1 \Leftrightarrow \text{WEAK COUPLING})$

THEN

$$x_1 \approx A \cos \omega t \cos \frac{\Delta \omega}{2} t$$

$$x_2 \approx A \sin \omega t \sin \frac{\Delta \omega}{2} t$$

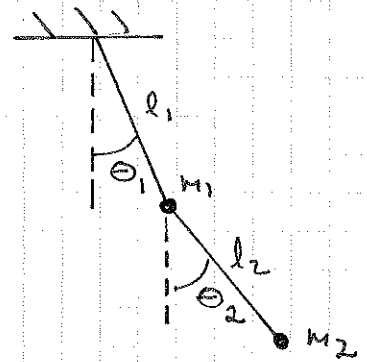
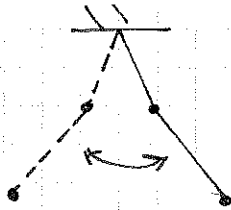
$$\omega = \frac{\omega_1 + \omega_2}{2} \quad \Delta \omega = \omega_2 - \omega_1$$



THE ENVELOPE OF THE MOTION IS THE SO-CALLED 'BEAT' PHENOMENON (OF FREQUENCY $\frac{\omega_1 - \omega_2}{2}$), PROMINENT WHENEVER OSCILLATORY MOTION OF NEARLY EQUAL FREQUENCIES IS COMBINED.

EXAMPLE 3 THE DOUBLE PENDULUM

FOR ARBITRARY l_1, l_2, m_1, m_2 WE HAVE TO CONFRONT THE PROBLEM THE HARD WAY. BUT WE KNOW ROUGHLY WHAT THE MODES WILL LOOK LIKE



TO DERIVE THE EQUATIONS OF MOTION WE USE LAGRANGE'S METHOD. THE HARDEST PART IS TO CHOOSE GOOD COORDINATES! THEY MUST BE INDEPENDENT, AND THERE MUST BE ONLY 2 (FOR MOTION IN A VERTICAL PLANE). TWO ANGLES SEEM APPROPRIATE. THE STANDARD TRICK IS TO MEASURE BOTH ANGLES FROM THE VERTICAL. (RATHER THAN θ_2 WITH RESPECT TO THE LINE OF ROD 1).

WE SIMPLIFY SLIGHTLY BY CONSIDERING ONLY THE CASE $l_1 = l_2 = l$.

$$L = T - V = T_1 + T_2 - V_1 - V_2$$

$$T_1 = \frac{1}{2} m_1 l^2 \dot{\theta}_1^2$$

$$V_1 = m_1 g l (1 - \cos \theta_1)$$

$$V_2 = m_2 g l (2 - \cos \theta_1 - \cos \theta_2)$$

FOR T_2 WE NOTE $x_2 = l(\sin \theta_1 + \sin \theta_2)$, $y_2 = -l(\cos \theta_1 + \cos \theta_2)$

$$\text{LEADING TO } T_2 = \frac{1}{2} m_2 l^2 (\dot{\theta}_1^2 + 2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2)$$

FOR SMALL OSCILLATIONS IT IS BEST TO MAKE THE SMALL ANGLE APPROXIMATION BEFORE CALCULATING THE EQUATIONS OF MOTION. BUT REMEMBER, YOU MUST KEEP TERMS UP TO 2ND ORDER IN THE LAGRANGIAN.

$$T_1 = \frac{1}{2} m_1 l^2 \dot{\theta}_1^2$$

$$V_1 \approx \frac{1}{2} m_1 g l \theta_1^2$$

NOTE THAT $\dot{\theta}_1$ & $\dot{\theta}_2$ ARE SMALL, SO $\cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \approx \dot{\theta}_1 \dot{\theta}_2$ TO 2ND ORDER

$$T_2 \approx \frac{1}{2} m_2 l^2 (\dot{\theta}_1^2 + 2 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \quad V_2 \approx \frac{1}{2} m_2 g l (\theta_1^2 + \theta_2^2)$$

$$L \approx \frac{1}{2} (m_1 + m_2) l^2 \dot{\theta}_1^2 + m_2 l^2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 l^2 \dot{\theta}_2^2 - \frac{1}{2} g l [(m_1 + m_2) \theta_1^2 + m_2 \theta_2^2]$$

$$\theta_1 \text{ eq} \Rightarrow (m_1 + m_2) l^2 \ddot{\theta}_1 + m_2 l^2 \ddot{\theta}_2 + g l (m_1 + m_2) \theta_1 = 0$$

$$\theta_2 \text{ eq} \Rightarrow m_2 l^2 \ddot{\theta}_1 + m_2 l^2 \ddot{\theta}_2 + g l m_2 \theta_2 = 0$$

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WE TRY SOLUTIONS $\theta_1 = A \cos \omega t$ $\theta_2 = B \cos \omega t$

$$(m_1 + m_2)(g - \omega^2 l) A - m_2 l \omega^2 B = 0$$

$$-m_2 l \omega^2 A + m_2(g - \omega^2 l) B = 0$$

[CHECK: THE MATRIX MUST BE SYMMETRIC!]

SETTING THE DETERMINANT TO ZERO, WE FIND

$$\omega^2 = \frac{g}{l} \frac{m_1 + m_2}{m_1} \left(1 \pm \sqrt{1 - \frac{m_1}{m_1 + m_2}} \right)$$

SOME SPECIAL CASES OF INTEREST ARE:

a) $m_1 = m_2 = m \Rightarrow \omega^2 = \frac{g}{l} (2 \pm \sqrt{2})$

$\omega_1 = \sqrt{\frac{1.7g}{l}}$ $\omega_2 \approx \sqrt{\frac{2.4g}{l}} \approx 2.3 \omega_1$

ALSO $\frac{B}{A} = \frac{l \omega^2}{g - \omega^2 l}$ SO FOR MODE 1, $\frac{B}{A} = \sqrt{2}$

WHILE FOR MODE 2, $\frac{B}{A} = -\sqrt{2}$

b) $m_1 = m$, $m_2 = \epsilon m \ll m$

$\omega^2 = \frac{g}{l} (1 \pm \sqrt{\epsilon})$ $\omega_{1,2} = \sqrt{\frac{g}{l}} \left(1 \pm \frac{\sqrt{\epsilon}}{2} \right)$

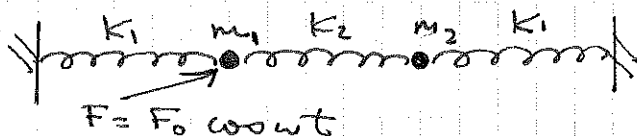
BECAUSE $\omega_1 \approx \omega_2$, BEATS SHOULD BE PROMINENT.

ALSO $B \approx \pm \frac{g}{l \sqrt{\epsilon}} A$, $|B| \gg |A|$

FORCED COUPLED OSCILLATIONS

THE CASE OF A SYSTEM OF COUPLED OSCILLATORS SUBJECT TO A SINUSOIDAL DRIVING FORCE IS EASY TO UNDERSTAND. WE EXPECT STEADY MOTION WITH ALL OSCILLATORS MOVING AT THE DRIVING FREQUENCY ω . RESONANCE SHOULD OCCUR WHENEVER ω IS A NORMAL FREQUENCY, RATHER THAN WHEN ω IS A NATURAL FREQUENCY OF SOME PARTICULAR SPRING IN THE SYSTEM.

EXAMPLE:



FROM P 150, THE EQUATION OF MOTION ARE $(m_1 = m_2 = m)$

$$m \ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = F_0 \cos \omega t$$

$$m \ddot{x}_2 + (K_1 + K_2)x_2 - K_2 x_1 = 0$$

WE TRY

$$x_1 = A_1 \cos \omega t \quad x_2 = A_2 \cos \omega t$$

WITH NO DAMPING
THERE IS NO PHASE
CHANGE.

$$\Rightarrow \begin{cases} (-\omega^2 M + k_1 + k_2) A_1 - k_2 A_2 = F_0 \\ -k_2 A_1 + (k_1 + k_2 - \omega^2 M) A_2 = 0 \end{cases}$$

$$\begin{aligned} \text{LET } \Delta &= \begin{vmatrix} k_1 + k_2 - \omega^2 M & -k_2 \\ -k_2 & k_1 + k_2 - \omega^2 M \end{vmatrix} = (k_1 + k_2 - M\omega^2)^2 - k_2^2 \\ &= (k_1 + 2k_2 - M\omega^2)(k_1 - M\omega^2) \\ &= M^2 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) \end{aligned}$$

WHERE $\omega_1 = \sqrt{\frac{k_1}{M}}$ AND $\omega_2 = \sqrt{\frac{k_1 + 2k_2}{M}}$ ARE THE NORMAL

FREQUENCIES OF THE SYSTEM.

$$\text{THEN } A_1 = \frac{\begin{vmatrix} F_0 & -k_2 \\ 0 & k_1 + k_2 - M\omega^2 \end{vmatrix}}{\Delta} = \frac{F_0}{M} \frac{\left(\frac{k_1 + k_2}{M} - \omega^2\right)}{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$A_2 = \frac{\begin{vmatrix} k_1 + k_2 - M\omega^2 & F_0 \\ -k_2 & 0 \end{vmatrix}}{\Delta} = \frac{F_0}{M} \frac{\left(\frac{k_2}{M}\right)}{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

LEADING TO RESONANCE AT EITHER NORMAL FREQUENCY!

THE NORMAL COORDINATES - A FORMAL PROCEDURE.

WE HAVE SEEN THAT THE PROBLEM OF COUPLED OSCILLATORS IS VERY EASY IF WE CAN GUESS THE RELATIONS AMONG THE COORDINATES IN EACH OF THE NORMAL MODES. THEN WE HAVE A SERIES OF ONE-DIMENSIONAL OSCILLATOR PROBLEMS - WHICH ARE READILY SOLVED. WE CAN SKETCH A FORMAL MATHEMATICAL PROCEDURE TO TAKE THE PLACE OF 'GUESSING'!

BASICALLY WE SEEK TO TRANSFORM OUR n COORDINATES q_j INTO A NEW SET OF COORDINATES $\eta_i = \eta_i(q_j)$

SUCH THAT ONLY ONE COORDINATE, η_i , VARIES PER NORMAL MODE. THAT IS, THE EQUATIONS OF MOTION IN TERMS OF THESE NEW COORDINATES ARE JUST $\ddot{\eta}_i + \omega_i^2 \eta_i = 0 \quad i = 1 \dots n$

WHERE THE ω_i ARE THE NORMAL FREQUENCIES. OF COURSE, WE CALL THE η_i THE NORMAL COORDINATES.

BUT HOW DO WE GET FROM THE q_j TO THE η_i ?

RECALL THAT THE EQUATIONS OF MOTION IN TERMS OF THE q_j ARE

$$\sum_i a_{ij} \ddot{q}_j + \sum_i k_{ij} q_j = 0 \quad (\text{P 148})$$

IN MATRIX TERMINOLOGY, WE NEED A COORDINATE TRANSFORMATION WHICH SIMULTANEOUSLY DIAGONALIZES THE TWO SYMMETRIC MATRICES a_{ij} AND k_{ij} .

WE NEED TO KNOW THAT ANY (REAL) SYMMETRIC MATRIX CAN BE DIAGONALIZED BY A SUITABLE LINEAR TRANSFORMATION

$$\text{i.e. } q'_i = \sum_j M_{ij} q_j$$

WE COULD DIAGONALIZE TWO MATRICES IN 3 STEPS.

1) DIAGONALIZE MATRIX a_{ij} BY A FIRST TRANSFORMATION.

$$a_{ij} \rightarrow \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix} \quad k_{ij} \rightarrow k'_{ij} \text{ STILL SYMMETRIC}$$

2) CHANGE THE SCALES OF THE NEW COORDINATES SO THAT EACH $\lambda_i \rightarrow 1$. I.E. LET $q''_i = \frac{1}{\lambda_i} q'_i$.

NOW a_{ij} HAS BEEN TRANSFORMED TO THE UNIT MATRIX $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, WHILE $k'_{ij} \rightarrow k''_{ij}$ IS STILL SYMMETRIC.

3) FIND A SECOND LINEAR TRANSFORMATION WHICH DIAGONALIZES MATRIX $k''_{ij} \rightarrow \begin{pmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \ddots \\ & & & \omega_n^2 \end{pmatrix}$. THE TRICK IS THAT

THE UNIT MATRIX REMAINS THE UNIT MATRIX UNDER THIS TRANSFORMATION! OUR NEW COORDINATES $q'''_j = \eta_j$ THE NORMAL COORDINATES!

SUCH A PROCEDURE COULD BE CARRIED OUT BY A COMPUTER FOR LARGE-SCALE PROBLEMS.

THE NORMAL COORDINATES CAN ALSO BE CONSTRUCTED OUT OF THE SOLUTIONS FOR THE NORMAL MODES FOUND BY THE MORE DIRECT METHOD. THIS PROCEDURE IS OUTLINED BY L&L IN SEC. 23.