# Ph 205 - Mechanics

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TEXT: LANDAU & LIFSHITZ ; MECHANICS, 3RD ED.

COURSE OUTLINE (ROUGHLY 1 WEEK PER TOPIC)

1. REVIEW OF ELEMENTARY MECHANICS
2. DEVELOPMENT OF A GENERAL METHOD
3. LAGRANGE'S EQUATIONS
4. CALCULUS OF VARIATIONS
5. EXAMPLES OF MOTION IN ONE AND TWO DIMENSIONS
6. MOTION UNDER A CENTRAL FORCE
7. COLLISIONS
8. OSCILLATIONS
9. ACCELERATED COORDINATE SYSTEMS
10. RIGID BODY ROTATIONS
11. OTHER ADVANCED METHODS
12. WAVE MOTION.

OTHER BOOKS WHICH MIGHT BE USEFUL ON OCCASION:

BARGER & OLSSON, CLASSICAL MECHANICS (AMUSING PROBLEMS)

GOLDSTEIN, CLASSIC MECHANICS, 2ND ED. (A LONG-TIME STANDARD)

SYMON, MECHANICS, 3RD ED
MARION, CLASSICAL DYNAMICS
} (TYPICAL TEXTS)

ROUTH - ANY BOOK
SOMMERFELD, MECHANICS
WHITTAKER ANALYTICAL DYNAMICS
} (IDIOSYNCRATIC CLASSICS!)

MACH THE SCIENCE OF MECHANICS (HISTORICAL PERSPECTIVE

ARNOLD. MATH. METH. OF CLASSICAL MECHANICS (A PEEK AT MODERN MECHANICS)

Ph 205 LECTURE 1

# REVIEW OF THE PRINCIPLES OF ELEMENTARY MECHANICS

IN BRIEF: $F = ma$!

## SOME HISTORY

WE BEGIN OUR STUDY WITH NEWTON'S LAWS, WHICH ARE THE RESULTS OF AT LEAST 2000 YEARS OF INTELLECTUAL STRUGGLE IN HIGHLY CONDENSED FORM. AS SUCH WE WILL MISS MUCH OF THE FLAVOR OF THE BATTLE FOR INSIGHT INTO THE REAL WORLD, SUBSTITUTING AN EFFORT TO MASTER TEXTBOOKS AND A RELATIVELY SMALL NUMBER OF IDEALIZED PROBLEMS.

IF YOU WISH TO READ MORE OF THE HISTORICAL PATH OF MECHANICS YOU MIGHT CONSULT:

'THE SCIENCE OF MECHANICS' BY ERNST MACH (PAPERBACK)
'HISTORY OF MECHANICS' BY RENE DUGAS
'THE COPERNICAN REVOLUTION' BY THOMAS KUHN (PAPERBACK)

'MECHANICS' IS STRICTLY SPEAKING THE SCIENCE OF MACHINES. ARISTOTLE & ARCHIMEDES HAD AN EXCELLENT UNDERSTANDING OF THE STATIC PROPERTIES OF THE MACHINES OF THEIR DAY. BUT THEY WERE DEFICIENT IN THE SUBJECT OF DYNAMICS — THE STUDY OF MOTION. FOR EXAMPLE, THEY HELD THAT A CONTINUAL APPLICATION OF FORCE IS NEEDED TO KEEP A BODY IN MOTION, AND THAT HEAVIER OBJECTS FALL FASTER THAN LIGHT ONES. THESE BELIEFS ARE READILY SUPPORTED BY A SOMEWHAT UNCRITICAL APPEAL TO EXPERIMENT.

ALTERNATIVE VIEWPOINTS WHICH BECAME THE FORE-RUNNERS OF THE NEWTONIAN SYNTHESIS APPEARED ONLY IN THE 1300'S. J. BURIDAN OF PARIS EXPOUNDED THE NOTION OF IMPETUS ABOUT 1350 — ROUGHLY THAT MOVING BODIES HAVE MOMENTUM WHICH MAINTAINS THEIR MOTION INDEPENDENT OF ANY FORCE. ABOUT THE SAME TIME ONE OR MORE PERSONS AT OXFORD SUGGESTED THAT ACCELERATION IS AN IMPORTANT CONCEPT. ABOUT 1380 ORESME, A STUDENT OF BURIDAN, SUGGESTED THE USE OF GRAPHICAL METHODS TO DETERMINE THE DISTANCE TRAVELLED BY AN OBJECT WITH UNIFORM VELOCITY, OR WITH UNIFORM ACCELERATION.

SO THE CONCEPTUAL FRAMEWORK FOR NEWTONIAN MECHANICS WAS SET ALREADY BY 1400. BUT INITIALLY THESE CONCEPTS WERE TAKEN AS MERELY A KIND OF MENTAL GAME TO SHOW THAT ALTERNATIVES TO ARISTOTLE WERE IMAGINABLE. IT TOOK 3 CENTURIES FOR THESE IDEAS TO BE COMPARED TO EXPERIENCE, AND TO BE DISTILLED INTO $F = ma$!

# Ph 205 LECTURE 1

[ BURIDAN HAS ACHIEVED RENEWED FAME AS THE FIRST EXPONENT OF SPONTANEOUS SYMMETRY BREAKING — THE IDEA THAT NATURE NEED NOT OBEY ALL MATHEMATICALLY CONCEIVABLE SYMMETRIES. HE GAVE AN EXAMPLE OF AN ASS BETWEEN TWO HAY STACKS. BASED ON LOGIC ALONE, THE ASS WOULD STARVE TO DEATH, UNABLE TO CHOOSE BETWEEN TWO EQUALLY DESIRABLE ALTERNATIVES. ]

## NEWTON'S LAWS

1. A MASS WILL REMAIN AT REST, OR IN A STATE OF UNIFORM MOTION, UNLESS ACTED UPON BY AN EXTERNAL FORCE.

THIS LAW DOES NOT EXPLAIN "MASS" OR "FORCE", BUT MAINTAINS THAT A MASS CAN HAVE MOTION INDEPENDENT OF THE CONTINUED ACTION OF AN OUTSIDE AGENT. BY IMPLICATION, CHANGES IN MOTION WILL BE ASCRIBED TO INTERACTIONS AMONG THINGS. THE REAL PROBLEM OF DYNAMICS IS NOT MOTION, BUT CHANGES IN MOTION. THE METHOD OF SOLUTION WILL BE 'DIVIDE AND CONQUER': BY SPLITTING THE PROBLEM INTO PROPERTIES OF MOTION, AND FORCES WHICH MODIFY MOTION.

2. $\bar{F} = d\bar{p}/dt$ WHERE $\bar{p} = m\bar{v}$ (NOT JUST $\bar{F} = m\bar{a}$)

NEWTON PROVIDED A MATHEMATICAL DESCRIPTION OF HOW FORCES MODIFY MOTION. ESSENTIALLY THIS COURSE WILL BE AN EXTENDED EXPLORATION OF THIS ONE RELATION.

THIS LAW CONTAINS THE EXTREMELY IMPORTANT OBSERVATION THAT FORCE IS A VECTOR QUANTITY. SEVERAL FORCES CAN BE ADDED LIKE VECTORS TO YIELD A SINGLE RESULTANT FORCE WHICH CAN THEN BE EQUATED TO $d\bar{p}/dt$.

3. TO EVERY FORCE THERE IS AN EQUAL AND OPPOSITE FORCE.

IF WE STUDY THE EFFECT OF THE REST OF THE UNIVERSE ON SOME OBJECT, THAT OBJECT ACTS BACK ON THE UNIVERSE IN A MANNER ANALYZABLE VIA THIS LAW.

PRACTICALLY AND HISTORICALLY THIS LAW PROVIDES A METHOD OF DEFINING, OR AT LEAST MEASURING, FORCES BY BRINGING THEM INTO BALANCE WITH THE FORCE OF GRAVITY. THIS IS POSSIBLE BECAUSE OF THE EMPIRICAL FACT THAT THE MASS IN $F = ma$ CAN ALWAYS BE SET EQUAL TO THE MASS IN THE GRAVITATIONAL FORCE LAW $F = mg$.

FINALLY, THE 3RD LAW ALLOWS THE STUDY OF EXTENDED

OBJECTS TO BE VASTLY SIMPLIFIED, BECAUSE 'INTERNAL FORCES' ALWAYS OCCUR IN PAIRS WHOSE NET EFFECT CANCELS FROM AN EXTERNAL VIEWPOINT.

WE NEXT SKETCH SOME PRINCIPLES OF THE MECHANICS OF A SINGLE POINT MASS, AND THEN SHOW HOW SIMILAR PRINCIPLES CAN BE ESTABLISHED FOR COLLECTIONS OF POINT PARTICLES.

IN EFFECT, WE SHOW THAT THE FORM OF THE LAWS OF MECHANICS IS INDEPENDENT OF WHAT EXACTLY HOLDS OBJECTS TOGETHER. THIS PROPERTY MAKES MECHANICS THE NATURAL STARTING POINT INTO INVESTIGATIONS OF THE NATURE OF THINGS. WE CAN OBTAIN A LARGE CLASS OF VALID RESULTS WITHOUT KNOWING ALL POSSIBLE DETAILS ABOUT MATTER. BUT CONVERSELY, WE CANNOT EXPECT MECHANICS TO PROVIDE A COMPLETE DESCRIPTION OF NATURE SINCE IT IS INSENSITIVE TO THE INNARDS OF THINGS.

IS IT TOO MUCH TO CONCLUDE THAT BECAUSE MECHANICS IS 'INCOMPLETE' THAT THERE MIGHT BE SOME REALM WHERE IT IS 'INCORRECT'?

Ph 205 LECTURE 1

## MECHANICS OF A SINGLE PARTICLE

FOR A SINGLE PARTICLE, $\bar{F}$ IS BY DEFINITION AN EXTERNAL FORCE. WE ALSO SUPPOSE THE MASS OF A POINT PARTICLE TO BE CONSTANT.

THEN INTEGRATION OF $\bar{F} = m\bar{a}$, COMBINED WITH A KNOWLEDGE OF THE POSITION $\bar{r}$ AND VELOCITY $\bar{v}$ AT SOME INITIAL TIME, GIVES A COMPLETE SOLUTION FOR THE MOTION. OF COURSE, $\bar{F}(t)$ MUST BE KNOWN.

AFTER SOLVING SEVERAL PROBLEMS ONE NOTES THE RECURRANCE OF SEVERAL 'CONSTANTS OF THE MOTION' UNDER CERTAIN CIRCUMSTANCES. THESE ARE EXTREMELY USEFUL IN THAT THE MOTION AT TWO TIMES CAN THEN BE SIMPLY RELATED WITHOUT PERFORMING THE INTEGRATION. TO GIVE THEM DUE HONOR THESE CONSTANTS OF THE MOTION ARE CALLED 'CONSERVED QUANTITIES' WHICH ARE SAID TO OBEY 'CONSERVATION LAWS'.

### 1. MOMENTUM: $\bar{P} = m\bar{v}$

NEWTON'S SECOND LAW STATES $d\bar{p}/dt = \bar{F}$

SO IF $\bar{F} = 0$, $\bar{p}$ = CONSTANT.

OF COURSE, A SINGLE PARTICLE PROBLEM WITH $\bar{F} = 0$ IS NOT MUCH OF A PROBLEM!

### 2. ANGULAR MOMENTUM: $\bar{L} = \bar{r} \times \bar{P}$

$\bar{r} \equiv$ POSITION OF THE PARTICLE WITH RESPECT TO SOME FIXED ORIGIN.

NOTE THAT $\bar{L}$ IS DEFINED WITH RESPECT TO A POINT NOT AN AXIS.

ALSO NOTE THAT A PARTICLE MOVING IN A STRAIGHT LINE HAS NON-ZERO ANGULAR MOMENTUM ABOUT ANY POINT NOT ON THAT LINE.

WE CAN EASILY GIVE AN EQUATION FOR CHANGES IN ANGULAR MOMENTUM:

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}(\bar{r} \times \bar{p}) = \frac{d\bar{r}}{dt} \times \bar{P} + \bar{r} \times \frac{d\bar{p}}{dt} = \bar{v} \times \bar{p} + \bar{r} \times \bar{F} = \bar{r} \times \bar{F}$$

SINCE $\bar{p} = m\bar{v}$ AND $\bar{v} \times \bar{v} = 0$.

WE DEFINE TORQUE $= \bar{N} = \bar{r} \times \bar{F}$ ABOUT THE ORIGIN (NOT ABOUT AN AXIS!).

HENCE $\dfrac{d\bar{L}}{dt} = \bar{N}$

CLEARLY IF $\bar{N} = 0$, $\bar{L}$ IS A CONSTANT.

NOTE THAT SINCE $\bar{L}$ IS A VECTOR IT IS POSSIBLE THAT SOME COMPONENTS REMAIN CONSTANT WHILE OTHERS VARY.

AN IMPORT CLASS OF PROBLEMS WITH $\bar{N} = 0$ IS THAT INVOLVING A CENTRAL FORCE, $\bar{F} = F\,\hat{r}$ $\left[\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \text{UNIT VECTOR}\right]$

THEN $\bar{L}$ = CONSTANT $\Rightarrow$ THE PATH OF THE PARTICLE SWEEPS OUT EQUAL AREAS IN EQUAL TIMES — KEPLER'S 2ND LAW.

## 3. WORK AND ENERGY

IF WE WISH TO EMPHASIZE THE MOTION OF A PARTICLE IN SPACE WITHOUT EXPLICIT MENTION OF TIME WE ARE LED TO DEFINE THE WORK DONE ON THE PARTICLE

$$W_{1,2} = \int_1^2 \bar{F}\cdot d\bar{s}$$

IN MOVING FROM POSITION 1 TO POSITION 2.

NEWTON'S 2ND LAW, FOR CONSTANT MASS, ALLOWS US TO REWRITE $\bar{F}\cdot d\bar{s}$ AS A PERFECT DIFFERENTIAL OF THE

KINETIC ENERGY = $T = \frac{1}{2} m v^2$

NAMELY $d\bar{s} = \bar{v}\,dt$ AND $\bar{F} = m\, d\bar{v}/dt$

SO $\bar{F}\cdot d\bar{s} = m\,\bar{v}\cdot \dfrac{d\bar{v}}{dt}\,dt = d\left(\frac{1}{2} m v^2\right) = dT$

HENCE $W_{12} = \int_1^2 dT = T_2 - T_1$

OFTEN THE WORK CAN BE CALCULATED WITHOUT KNOWING WHEN THE PARTICLE WAS AT A PARTICULAR POSITION. THIS ALLOWS $v$ TO BE CALCULATED AS A FUNCTION OF POSITION $\bar{r}$ WITHOUT KNOWING $\bar{r}(t)$.

IN MANY INTERESTING CASES IT HAPPENS THAT $\int_1^2 \bar{F}\cdot d\bar{s}$ IS INDEPENDENT OF THE PATH TAKEN BETWEEN 1 AND 2. A FORCE WHICH HAS THIS PROPERTY IS CALLED CONSERVATIVE.

EXAMPLES OF CONSERVATIVE FORCES ARE GRAVITY, AN IDEAL SPRING FORCE, THE ELECTROSTATIC FORCE. FROM THE POINT OF VIEW OF MECHANICS, FRICTION IS A NON-CONSERVATIVE FORCE.

FOR A CONSERVATIVE FORCE THE INTEGRAL AROUND ANY CLOSED LOOP VANISHES: $\oint \bar{F}\cdot d\bar{s} = 0.$

EXPERTS IN VECTOR CALCULUS WILL RECALL STOKES' THEOREM:

$$\oint \bar{F}\cdot d\bar{s} = \int_{\text{SURFACE OF LOOP}} (\bar{\nabla}\times\bar{F})\cdot d\bar{A}$$

THEN IF $\oint \bar{F}\cdot d\bar{s} = 0$ FOR ANY LOOP, WE MUST HAVE $\bar{\nabla}\times\bar{F} = 0$.

BUT IF $\bar{\nabla}\times\bar{F} = 0$ THEN THERE EXISTS A SCALAR FUNCTION $V(\bar{r})$ SUCH THAT $\bar{F} = -\bar{\nabla}V$

$V(r)$ IS THE <u>POTENTIAL ENERGY</u>

OR, WE MAY PROCEED IN A MORE STRAIGHTFORWARD, IF LESS ELEGANT MANNER. SINCE $\int_1^2 \bar{F}\cdot d\bar{s}$ IS INDEPENDENT OF THE PATH, WE MAY UNIQUELY DEFINE $V(\bar{r}) = -\int_{\bar{r}_0}^{\bar{r}} \bar{F}\cdot d\bar{s}$

ONCE A REFERENCE POINT $\bar{r}_0$ IS CHOSEN, FOR WHICH $V(\bar{r}_0) = 0$.

IF A DIFFERENT REFERENCE POINT $\bar{r}_0'$ WERE USED,

$$V'(\bar{r}) = -\int_{r_0'}^{\bar{r}} \bar{F}\cdot d\bar{s} = -\int_{\bar{r}_0}^{\bar{r}} \bar{F}\cdot ds - \int_{\bar{r}_0'}^{\bar{r}_0} \bar{F}\cdot d\bar{s} = V(\bar{r}) + \text{CONSTANT}.$$

HENCE THE POTENTIAL IS DEFINED ONLY UP TO AN ADDITIVE CONSTANT — IN THAT WE ARE FREE TO CHANGE OUR POINT OF REFERENCE.

YOU MAY WISH TO CONVINCE YOURSELF BY DIRECT DIFFERENTIATION THAT $\bar{F} = -\bar{\nabla}V$.

OR NOTE THAT $-\int_{\bar{r}_0}^{\bar{r}} \bar{F}\cdot d\bar{s} = \int_{\bar{r}_0}^{\bar{r}} \bar{\nabla}V\cdot d\bar{s} = \int_{\bar{r}_0}^{\bar{r}} \frac{\partial V}{\partial s}\, ds = V(\bar{r}) - V(\bar{r}_0) = V(\bar{r})$

FINALLY, OUR RELATION $W_{12} = T_2 - T_1$

CAN BE EXTENDED SINCE $W_{12} = \int_1^2 \bar{F}\cdot d\bar{s} = V_1 - V_2$.

HENCE $V_1 + T_1 = V_2 + T_2$.

IF WE DEFINE $\underline{E = T + V = \text{TOTAL ENERGY}}$ THEN $E$ IS A CONSERVED QUANTITY (FOR CONSERVATIVE FORCES).

THIS LEADS TO THE USUAL INTERPRETATION OF $V$ AS THE 'POTENTIAL ENERGY' STORED IN A CERTAIN CONFIGURATION OF THE SYSTEM DUE TO THE WORK DONE IN ARRANGING THAT CONFIGURATION.

IT IS INSTRUCTIVE TO VERIFY THAT $dE/dt = 0$.

$E = T + V$ SO $\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt}$

$\frac{dT}{dt} \equiv \underline{\text{POWER}} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\bar{v}\cdot\frac{d\bar{v}}{dt} = \bar{F}\cdot\bar{v}$ (A USEFUL RELATION)

BY OUR ASSUMPTION OF INDEPENDENCE OF THE PATH, $V$ CANNOT DEPEND ON $t$ EXPLICITLY. i.e. $V = V(\bar{r})$ NOT $V(\bar{r}, t)$.

BUT SINCE THE PATH IS $\bar{r} = \bar{r}(t)$, THEN $V = V(\bar{r}(t))$

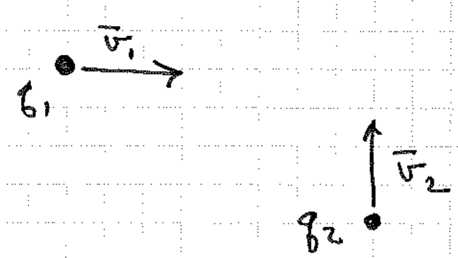
SO $\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} = \bar{\nabla}V\cdot\bar{v}$

ALTOGETHER $\frac{dE}{dt} = \bar{F}\cdot\bar{v} + \bar{\nabla}V\cdot v = \left(\bar{F} + \bar{\nabla}V\right)\cdot\bar{v} = 0$

SINCE $\bar{F} = -\bar{\nabla}V$.

THESE CONSERVATION LAWS ARE SO MARVELOUS THAT WE HAVE COME TO BELIEVE IN THEM RATHER STRONGLY. WE EXPECT ANY FUNDAMENTAL INTERACTION BETWEEN TWO PARTICLES TO OBEY ALL 3 CONSERVATION LAWS, AS WELL AS OTHER CONSERVATION LAWS PARTICULAR TO THE TYPE OF FORCE – SUCH AS CHARGE CONSERVATION IN ELECTRICITY. IF EXPERIMENT FAILS TO CONFIRM THE LAWS THE COMMON TENDENCY IS TO SUSPECT THE EXPERIMENTS. IF NO SOURCE OF ERROR IS FOUND THEN ONE TENDS TO CLAIM THE INTERACTION ACTUALLY INVOLVED AN ADDITIONAL PARTICLE WHOSE PRESENCE RESTORES THE CONSERVATION LAWS. THUS IN ASTRO PHYSICS THE DISCOVERY OF THE OUTER PLANETS, NEUTRON STARS, AND PERHAPS NOW BLACK HOLES. IN SUB-ATOMIC PHYSICS THE NEUTRINO WAS POSTULATED TO PATCH UP ENERGY CONSERVATION IN BETA DECAY.

YOU MAY WISH TO CONTEMPLATE THE FOLLOWING PARADOX. CONSIDER THE ELECTRIC AND MAGNETIC FORCES OF TWO MOVING CHARGED PARTICLES ON EACH OTHER, WHEN THE VELOCITIES ARE NOT PARALLEL.




IN GENERAL THIS APPEARS TO VIOLATE NEWTON'S 3RD LAW AS WELL AS ALL 3 CONSERVATION LAWS.

# Ph 205 Lecture 1

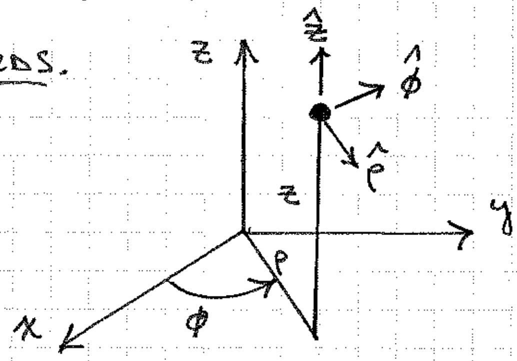
## Kinematics of Particle Motion

We make a short digression into solid geometry to give a description of the motion of a particle in other than rectangular coordinates.

### 1. Cylindrical Coords.

$\bar{r} = (\rho, \phi, z)$

$x = \rho \cos\phi$
$y = \rho \sin\phi$




$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$
$\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$

We want the components of velocity and acceleration in cylindrical coords.

$$\bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$
$$= (\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi})(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi) + (\dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi})(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi) + \dot{z}\hat{z}$$
$$= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \quad \text{OR} \quad \underline{\bar{v} = (\dot{\rho}, \rho\dot{\phi}, \dot{z})}$$

$$\bar{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$
$$= (\ddot{\rho}\cos\phi - 2\dot{\rho}\sin\phi\,\dot{\phi} + \rho\sin\phi\,\ddot{\phi} - \rho\cos\phi\,\dot{\phi}^2)(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi)$$
$$+ (\ddot{\rho}\sin\phi + 2\dot{\rho}\cos\phi\,\dot{\phi} + \rho\cos\phi\,\ddot{\phi} - \rho\sin\phi\,\dot{\phi}^2)(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi) + \ddot{z}\hat{z}$$
$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z} \quad \text{or} \quad \underline{\bar{a} = (\ddot{\rho} - \rho\dot{\phi}^2, \rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}, \ddot{z})}$$

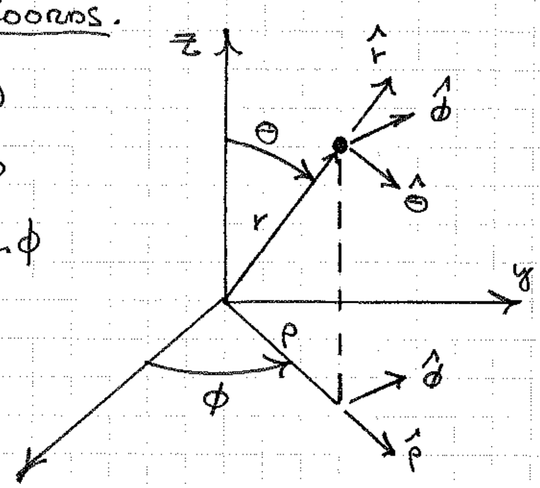
### 2. Spherical Coords.

$\bar{r} = (r, \theta, \phi)$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$




$\hat{\rho} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$

$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$
$\quad = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

AFTER SOME ALGEBRA WE FIND THAT

$$\bar{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2)\hat{\theta}$$
$$+ (r\sin\theta\,\ddot{\phi} + 2\dot{r}\sin\theta\,\dot{\phi} + 2r\cos\theta\,\dot{\theta}\dot{\phi})\hat{\phi}$$

WE VERIFY THAT FOR MOTION IN A PLANE, SPHERICAL COORDS CAN BE REDUCED TO POLAR COORDS IN 2 WAYS:

a. SET $\theta = \pi/2 \Rightarrow$ MOTION IN THE $x$-$y$ PLANE

$$\bar{v} \to (\dot{r},\ 0,\ r\dot{\phi})$$

$$\bar{a} \to (\ddot{r} - r\dot{\phi}^2,\ 0,\ r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

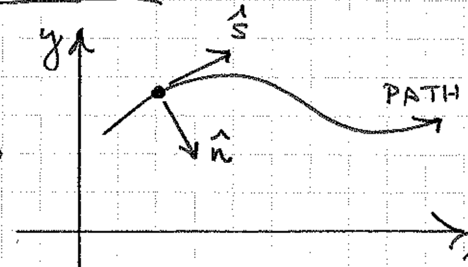
b. SET $\phi =$ CONSTANT $\Rightarrow$ MOTION IN A PLANE CONTAINING THE $z$ AXIS.

$$\bar{v} \to (\dot{r},\ r\dot{\theta},\ 0)$$

$$\bar{a} \to (\ddot{r} - r\dot{\theta}^2,\ r\ddot{\theta} + 2\dot{r}\dot{\theta},\ 0)$$

## 3. ARC COORDINATES, OR INTRINSIC COORDINATES

FIRST WE CONSIDER MOTION IN A PLANE.



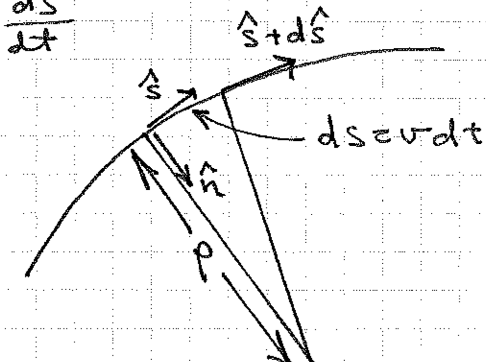

LET $S$ MEASURE DISTANCE ALONG THE PATH,

$S =$ ARC LENGTH

WE SET UP A LOCAL COORDINATE SYSTEM WITH UNIT VECTORS

$\hat{s} =$ TANGENT AND $\hat{n} =$ NORMAL

THE MAGNITUDE OF THE VELOCITY IS JUST $v = \dfrac{dS}{dt}$




SO $\bar{v} = v\,\hat{s}$

THEN $\bar{a} = \dfrac{d\bar{v}}{dt} = \dot{v}\,\hat{s} + v\dfrac{d\hat{s}}{dt}$

FROM THE PICTURE, $d\hat{s} = \hat{n}\,\dfrac{v\,dt}{\rho}$

WHERE $\rho =$ RADIUS OF CURVATURE OF THE PATH.

HENCE $\bar{a} = \dot{v}\,\hat{s} + \dfrac{v^2}{\rho}\,\hat{n}$

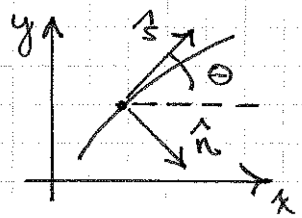
 

THUS THE ACCELERATION CAN BE THOUGHT OF AS CONSISTING OF 2 PARTS:

1. ACCELERATION ALONG THE DIRECTION OF MOTION, WHICH CHANGES THE SPEED, BUT NOT THE DIRECTION.

2. CIRCULAR MOTION ABOUT AN INSTANTANEOUS CENTER OF CURVATURE, WHICH CHANGES THE DIRECTION, BUT NOT THE SPEED. THIS GIVES A COMPONENT OF ACCELERATION TRANSVERSE TO THE VELOCITY, WHOSE MAGNITUDE IS $v^2/\rho$, FAMILIAR FROM UNIFORM CIRCULAR MOTION.

ANOTHER DERIVATION OF THE ACCELERATION:




$$a_s = a_x \cos\theta + a_y \sin\theta$$

$$a_n = a_x \sin\theta - a_y \cos\theta$$

$$\bar{a} = a_s \hat{s} + a_n \hat{n}$$

BUT $\cos\theta = v_x/v$, $\sin\theta = v_y/v$

SO $$a_s = \dot{v}_x \frac{v_x}{v} + \dot{v}_y \frac{v_y}{v} = \frac{1}{v} \cdot \frac{1}{2} \frac{dv^2}{dt} = \frac{dv}{dt}$$

$$a_n = \frac{\dot{v}_x v_y - \dot{v}_y v_x}{v} = v^2 \frac{(\ddot{x}\dot{y} - \ddot{y}\dot{x})}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = v^2/\rho$$

USING A RESULT FROM CALCULUS FOR THE RADIUS OF CURVATURE.

## MOTION IN 3 DIMENSIONS

THE ACCELERATION OF A PARTICLE IS DETERMINED BY 3 OBSERVATIONS OF ITS MOTION, SEPARATED BY VERY SHORT TIME INTERVALS. THE 3 INSTANTANEOUS POSITIONS OF THE PARTICLE DURING THIS OBSERVATION CAN BE USED TO DETERMINE A PLANE, WHICH CAN BE THOUGHT OF AS THE INSTANTANEOUS, OR 'OSCULATING' PLANE OF THE MOTION.

A THIRD UNIT VECTOR BESIDES $\hat{s}$ AND $\hat{n}$, CALLED THE BINORMAL $= \hat{b} = \hat{s} \times \hat{n}$ COMPLETES OUR LOCAL COORD. SYSTEM.

IN THIS SYSTEM $(s, n, b)$ WE HAVE AT ONCE

$$\bar{v} = (v, 0, 0)$$

$$\bar{a} = (\dot{v}, v^2/\rho, 0)$$

# Ph 205 LECTURE 1

## FRENET FORMULAE — A CULTURAL DIGRESSION.

RATHER THAN DESCRIBE THE MOTION OF A PARTICLE BY ITS HISTORY: $\bar{r}(t)$, $\bar{v}(t)$, $\bar{a}(t)$, WE SOMETIMES JUST CONSIDER THE ORBIT OR PATH OF THE PARTICLE, WITHOUT CARING WHEN IT WAS WHERE.

IF $S$ MEASURES THE ARC LENGTH, WE NOW CONSIDER $\bar{r}(S)$. WE WANT TO DESCRIBE CHANGES IN THE SHAPE OF THE ORBIT.

WE SAW THAT $\bar{v} = v\hat{s}$, OR $\frac{d\bar{r}}{dt} = \frac{dS}{dt}\hat{s}$, SO $\underline{\frac{d\bar{r}}{dS} = \hat{s}}$

THE UNIT NORMAL TO THE PATH IS JUST $\hat{n} = \frac{d\hat{s}}{dS} \Big/ \left|\frac{d\hat{s}}{dS}\right|$

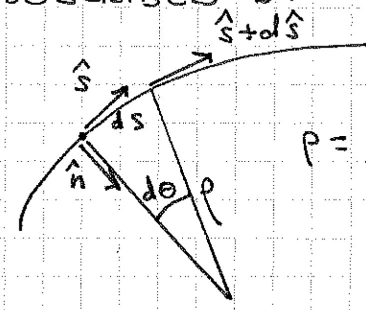
NOTE THAT $\hat{s}\cdot\hat{n} \sim \hat{s}\cdot\frac{d\hat{s}}{dS} = \frac{1}{2}\frac{d\hat{s}^2}{dS} = 0$ SINCE $\hat{s}^2 = 1$ ALWAYS

I.E. THE NORMAL IS REALLY PERPENDICULAR TO $\hat{s}$.

THE BINORMAL IS JUST $\hat{b} = \hat{s}\times\hat{n}$ AS BEFORE.

THE CHANGING SHAPE OF THE CURVE IS DESCRIBED BY

$\frac{d\hat{s}}{dS}$, $\frac{d\hat{n}}{dS}$ AND $\frac{d\hat{b}}{dS}$




$\rho = \frac{ds}{d\Theta}$

FROM THE PICTURE: $\frac{d\hat{s}}{dS} = \frac{\hat{n}}{\rho}$

AS THE PARTICLE TRAVELS DISTANCE $dS$, THE OSCULATING PLANE OF THE ORBIT CAN, IN GENERAL, TWIST ABOUT $\hat{s}$ BY SOME ANGLE $d\phi$. WE DEFINE THE RADIUS OF TORSION TO BE

$$\sigma = \frac{dS}{d\phi} \qquad (\sigma \text{ CAN BE NEGATIVE})$$

THEN $\frac{d\hat{b}}{dS} = -\frac{\hat{n}}{\sigma}$ (WHICH DEFINES THE SIGN OF $\sigma$)

FINALLY, SINCE $\hat{n} = \hat{b}\times\hat{s}$

$$\frac{d\hat{n}}{dS} = \frac{d\hat{b}}{dS}\times\hat{s} + \hat{b}\times\frac{d\hat{s}}{dS} = -\frac{\hat{n}\times\hat{s}}{\sigma} + \frac{\hat{b}\times\hat{n}}{\rho} = \frac{\hat{b}}{\sigma} - \frac{\hat{s}}{\rho}$$

THESE 3 RESULTS ARE THE SO-CALLED FRENET FORMULAE.

$$\frac{d\hat{s}}{ds} = \frac{\hat{n}}{\rho} \, , \quad \frac{d\hat{n}}{ds} = \frac{\hat{b}}{\sigma} - \frac{\hat{s}}{\rho} \, , \quad \frac{d\hat{b}}{ds} = -\frac{\hat{n}}{\sigma}$$

NOTE THAT IF WE DEFINE A VECTOR $\bar{\Omega} = \frac{\hat{s}}{\sigma} + \frac{\hat{b}}{\rho}$

THEN $\frac{d\hat{s}}{ds} = \bar{\Omega} \times \hat{s}$ , $\frac{d\hat{n}}{ds} = \bar{\Omega} \times \hat{n}$ , $\frac{d\bar{b}}{ds} = \bar{\Omega} \times \hat{b}$

SINCE $\hat{s}, \hat{n}, \hat{b}$ FORM A RIGID TRIAD OF VECTORS, WE CAN ANTICIPATE A RESULT TO BE DISCUSSED LATER: ANY MOTION OF A RIGID BODY CAN BE DECOMPOSED INTO A TRANSLATION AND A ROTATION.