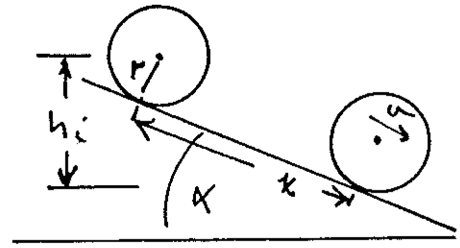
Ph205 Solutions to Final Exam Jan 1989

(1) Unrolling Tape



We follow the hint to work b) before a)

b) The wheel was invented to beat friction, so we don't expect friction to eat any energy as the tape unwinds. But there must be friction (or a nail!) to keep the tape from sliding.

Anyway, no energy loss to friction $\Rightarrow \Delta KE = \Delta PE$

$KE = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$ M = mass of tape left on bobbin
$I = M r^2$ = moment of inertia of tape on bobbin

$M_{LEFT} = M_0 \left(1 - \frac{x}{\ell}\right) \equiv M_0 (1-u)$ where M_0 = total mass of tape
ℓ = total length of tape

Also $\omega = v/r$

so $KE = \frac{M_0}{2}(1-u)v^2 + \frac{1}{2} M_0 (1-u) r^2 \frac{v^2}{r^2} = M_0(1-u)v^2$

Meanwhile, ΔPE = change in gravitational potential energy

$$= PE_i - PE_f = M_0 g \underbrace{(x \sin\alpha + r\cos\alpha)}_{h_i} - \underbrace{M g r \cos\alpha}_{PE_f \text{ of bobbin}} - \underbrace{M_0 u g \tfrac{x}{2}\sin\alpha}_{PE_f \text{ of tape on incline}}$$

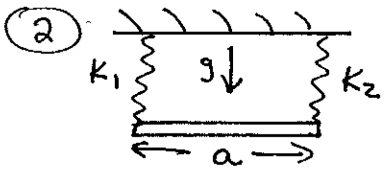
$$= M_0 g \left[\left(1 - \frac{u}{2}\right) \underset{= u\ell}{x} \sin\alpha + r u \cos\alpha\right]$$

So $$\boxed{v^2 = g \frac{u}{1-u}\left[r\cos\alpha + \left(1 - \frac{u}{2}\right)\ell \sin\alpha\right]}$$

Note: As $u \to 1$, $v \to \infty$!!

a) Although in b) we assumed conservation of energy, we are led to a peculiar conclusion as the tape runs out: the end of the tape moves very fast at the last moment. This end contains the 'missing' energy – the total ΔPE. At the last moment the end comes down like a whip & dissipates the energy in an inelastic collision with the incline – but only at the last moment; otherwise the tape unwinds smoothly while conserving energy.

(2)

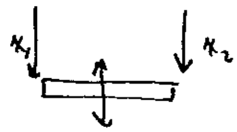


PH 205 FINAL EXAM

UNDER THE RESTRICTIONS THAT WE HAVE MOTION IN A VERTICAL PLANE, AND THAT THE ENDS MOVE (ESSENTIALLY) VERTICALLY, THE BAR HAS ONLY 2 DEGREES OF FREEDOM $\Rightarrow$ 2 MODES.

a) $K_1 = K_2$ WE USE ELEMENTARY METHODS TO HAVE A CHECK ON USE OF LAGRANGE'S METHOD FOR b)

MODE 1:

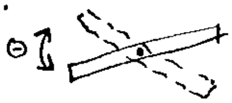


$x_1 = x_2 \equiv x \Leftrightarrow$ NO ROTATION

CM MOTION: $m\ddot{x} = -(k_1+k_2)x = -2Kx$

$$\boxed{\omega_1 = \sqrt{\frac{2K}{m}}}$$

MODE 2:



$x_1 = -x_2 \Leftrightarrow$ C.M. AT REST $\Leftrightarrow$ ROTATION

$I\ddot{\theta} = N_{\text{ABOUT C.M.}} = -2Kx_1\left(\frac{a}{2}\right) = -\frac{Ka^2}{2}\theta$ USING $\theta = \frac{x_1}{a/2}$

$I_{cm} = \frac{1}{12}ma^2$ SO $\ddot{\theta} = -\frac{6K}{m}\theta \Rightarrow \boxed{\omega_2 = \sqrt{\frac{6K}{m}}}$

b) $K_1 \neq K_2$ FIRST, THE STRETCHES AT EQUILIBRIUM:

NO TORQUE $\Rightarrow F_1 = F_2$

CM AT REST $\Rightarrow F_1 + F_2 = mg$

$F_1 = k_1x_1$; $F_2 = k_2x_2$ SO $k_1x_1 = k_2x_2$ & $k_1x_1 + k_2x_2 = mg$

$$\Rightarrow \boxed{x_1 = \frac{mg}{2k_1} \;;\; x_2 = \frac{mg}{2k_2}}$$

WHERE x_1 & x_2 ARE MEASURED FROM THE ENDS OF THE UNSTRETCHED SPRINGS

LET $q_1 \equiv x_1 - \frac{mg}{2k_1}$ $\quad q_2 \equiv x_2 - \frac{mg}{2k_2}$

THEN $V = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2q_2^2$ [THE PIECE mgh_{cm} IS ABSORBED IN THE DEFINITION OF THE q_i !!]

$$T = T_{cm} + T_{\text{ROTATION}} = \frac{1}{2}m\left(\frac{\dot{q}_1+\dot{q}_2}{2}\right)^2 + \frac{1}{2}\underbrace{\frac{1}{12}ma^2}_{I}\underbrace{\left(\frac{\dot{q}_1-\dot{q}_2}{a}\right)^2}_{\omega^2}$$

$\leftarrow$ NOT $2a$!

$$= \frac{m}{2}\left[\frac{\dot{q}_1^2}{3} + \frac{\dot{q}_1\dot{q}_2}{3} + \frac{\dot{q}_2^2}{3}\right]$$

LAGRANGE $\Rightarrow$ EQ. OF MOTION:

$$\frac{m\ddot{q}_1}{3} + \frac{m\ddot{q}_2}{6} = -k_1q_1$$

$$\frac{m\ddot{q}_1}{6} + \frac{m\ddot{q}_2}{3} = -k_2q_2$$

TRY AN OSCILLATORY SOLUTION: $q_1 = A\cos\omega t$ $\quad q_2 = B\cos\omega t$

$$\Rightarrow \left(\frac{k_1}{m} - \frac{\omega^2}{3}\right)A - \frac{\omega^2}{6}B = 0$$

$$-\frac{\omega^2}{6}A + \left(\frac{k_2}{m} - \frac{\omega^2}{3}\right)B = 0$$

SET DETERMINANT TO ZERO $\Rightarrow \frac{K_1K_2}{m^2} - \frac{K_1+K_2}{3m}\omega^2 + \frac{\omega^4}{12} = 0$

$$\omega^4 - 4\frac{K_1+K_2}{m}\omega^2 + 12\frac{K_1K_2}{m^2} = 0 \Rightarrow \boxed{\omega^2 = 2\frac{K_1+K_2}{m} \pm \frac{1}{m}\sqrt{4(K_1+K_2)^2 - 12K_1K_2}}$$

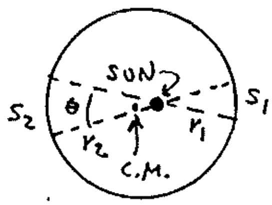
CHECK: $K_1 = K_2 \equiv K \Rightarrow \omega^2 = \frac{4K}{m} \pm \frac{2K}{m} = \left(\frac{2K}{m}, \frac{6K}{m}\right)$ SO O.K. !

(3) RINGWORLD

RINGWORLD IS SUCH A BAD IDEA THAT PERHAPS A SIMPLE ARGUMENT WILL SUFFICE TO DISPOSE OF IT. BUT WE WILL FOLLOW UP WITH A DETAILED ARGUMENT.

CASE I. C.M. HAS NO ANGULAR MOMENTUM ABOUT THE SUN $\Rightarrow$ MOVES IN STRAIGHT LINE

? IF C.M. MOVES OFF THE SUN, WILL IT BE PULLED BACK ??



F ON ARC $S_1 \sim \frac{S_1}{r_1^2}$ TO LEFT BUT $S_1 = r_1\theta$ SO $F_1 \sim \frac{\theta}{r_1}$

F ON ARC $S_2 \sim \frac{S_2}{r_2^2}$ TO RIGHT $S_2 = r_2\theta$ SO $F_2 = \frac{\theta}{r_2}$

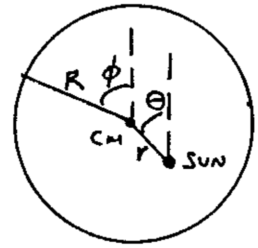
F_{TOTAL} TO LEFT IS $\theta\left(\frac{1}{r_1} - \frac{1}{r_2}\right) > 0 \Rightarrow$ C.M. PULLED AWAY FROM SUN $\Rightarrow$ UNSTABLE (EVEN A GRASSHOPPER'S JUMP WOULD SET THIS DISASTER IN MOTION !!)

CASE II. ADD ANGULAR MOMENTUM OF C.M. ABOUT SUN, THIS IS EVEN WORSE, BECAUSE IN THE ROTATING FRAME THERE IS AN ADDITIONAL FORCE (CENTRIFUGAL FORCE) DRIVING THE CM AWAY FROM THE SUN. $\Rightarrow$ UNSTABLE

[DO YOU REALLY WANT TO TRUST THE SPACE PROGRAM TO SCIENCE FICTION WRITERS? WOULD A 'SPHERE WORLD' BE ANY BETTER]

LET'S VERIFY THIS WITH A HIGH-POWERED TECHNIQUE: THE EFFECTIVE-POTENTIAL METHOD.

$$T = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \tfrac{1}{2}I\dot\phi^2$$

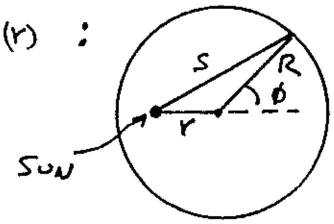


BUT $V = V(r)$ ONLY

$\Rightarrow L_\theta = mr^2\dot\theta = \text{CONST}$; $L_\phi = I\dot\phi = \text{CONST}$

$$E = T + V = \text{CONST} = \tfrac{1}{2}m\dot r^2 + \underbrace{\frac{L_\theta^2}{2mr^2} + V(r)}_{V_{eff}(r)} + \underbrace{\tfrac{1}{2}I\dot\phi^2}_{\text{CONST}}$$

FOR $V(r)$:



$$V(r) = -\frac{GMm}{2\pi}\int d\phi \, \frac{1}{S(\phi)}$$

$$S = \sqrt{R^2 + r^2 + 2rR\cos\phi} \Rightarrow \text{ELLIPTIC INTEGRAL!}$$

$$= R\sqrt{1 + \frac{r^2}{R^2} + \frac{2r}{R}\cos\phi}$$

EXPAND $\frac{1}{S}$ ASSUMING $r/R \ll 1$. CAREFUL: WILL NEED TO GO TO 2ND ORDER AS $\int \cos\phi \, d\phi = 0$

IF $f = \frac{1}{\sqrt{1+x}}$ THEN $f' = -\frac{1}{2}\frac{1}{(1+x)^{3/2}}$ $\quad f'' = +\frac{3}{4}\frac{1}{(1+x)^{5/2}}$

SO $f(x) \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots = 1 - \frac{x}{2} + \frac{3}{8}x^2$

FOR US, $x = \frac{r^2}{R^2} + \frac{2r}{R}\cos\phi$, SO $\frac{R}{S} \approx 1 - \frac{r^2}{2R^2} - \frac{r}{R}\cos\phi + \frac{3}{8}\left(\frac{4r^2}{R^2}\cos^2\phi + O\frac{r^3}{R^3} \cdots\right)$

$$= 1 - \frac{r}{R}\cos\phi - \frac{r^2}{R^2}\left(\frac{1}{2} - \frac{3}{2}\cos^2\phi\right)$$

AND $V(r) = -\frac{GMm}{R}\left(1 - \frac{r^2}{R}\left(\frac{1}{2} - \frac{3}{4}\right)\right) = -\frac{GMm}{R}\left(1 + \frac{1}{4}\frac{r^2}{R^2}\right)$

$V(r)$ IS PARABOLIC <u>DOWNWARD</u> !!

CASE I. $L_\theta = 0$, $V_{eff} = V(r)$, SO $r = 0$ IS AN UNSTABLE EQUILIBRIUM!

CASE II $L_\theta \neq 0$ $\quad V_{eff} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{R}\left(1 + \frac{1}{4}\frac{r^2}{R^2}\right)$

WHERE IS EQUILIBRIUM? $\quad \frac{dV_{eff}}{dr} = -\frac{L^2}{mr} - \frac{GMmr}{2R^3}$

$\Rightarrow$ NO EQUILIBRIUM VALUE OF r !!

$\Rightarrow$ WILDLY UNSTABLE!

PH 205 FINAL EXAM

(4) THE STRANGE INSIGHT OF SCHRÖDINGER IS <u>NOT</u> TO TRY TO SOLVE (1'') DIRECTLY, BUT TO USE IT AS THE FUNCTIONAL ARGUMENT OF A VARIATIONAL PROBLEM

$$(1'') \Rightarrow \quad f(\psi, \psi') \equiv \left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{2m}{K^2}\left(E - V(x)\right)\psi^2$$

WE WANT $\delta I = \delta \int f(\psi, \psi')\, dx = 0$ UNDER VARIATIONS IN ψ

THIS IS THE STANDARD EULER-LAGRANGE PROBLEM, WHOSE SOLUTION OBEYS LAGRANGE'S EQUATIONS: $\dfrac{d}{dx}\dfrac{\partial f}{\partial \psi'} = \dfrac{\partial f}{\partial \psi}$

HENCE $\boxed{\dfrac{\partial^2 \psi}{\partial x^2} = -\dfrac{2m}{K^2}\left(E - V(x)\right)\psi}$

FOR $V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{ELSEWHERE} \end{cases}$

WE SEE THAT OUTSIDE $(0, a)$, ψ MUST VANISH FOR FINITE E, AS V IS INFINITE THERE.

ON $(0, a)$ WE HAVE $\psi'' = -\dfrac{2mE}{K^2}\psi$

$$\Rightarrow \psi = A \cos\sqrt{\frac{2mE}{K^2}}\, x + B \sin\sqrt{\frac{2mE}{K^2}}\, x$$

TO BE CONTINUOUS AT $x = 0$ AND a, ψ MUST VANISH THERE

$$\Rightarrow A = 0, \quad \text{AND} \quad \sqrt{\frac{2mE}{K^2}} = \frac{n\pi}{a}$$

$$\Rightarrow \boxed{E = \frac{K^2}{2m}\left(\frac{n\pi}{a}\right)^2}$$

IF YOU BELIEVE $E = \dfrac{p^2}{2m}$ STILL HOLDS, THEN $p = \dfrac{n\pi K}{a}$

AND $pa = n\pi K$ SO $\boxed{pa\big|_{MIN} = \pi K = \dfrac{h}{2}}$ FOLLOW SCHRÖDINGER THAT $K = \dfrac{h}{2\pi}$