Ph 205 Final Exam

Jan 18, 1989

7:30 PM

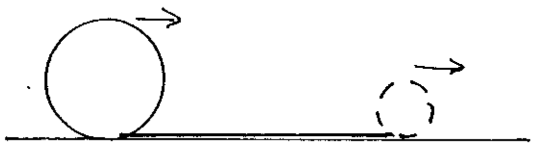
Time limit: 3 hours

The exam is closed book, closed note, etc.

Do all work you want graded in the exam
booklets provided

The exam consists of 4 problems each worth 10 points.

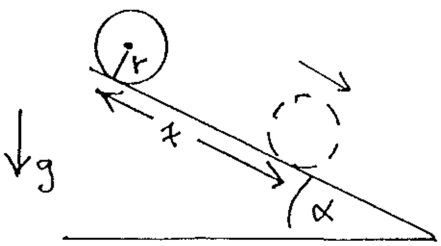
① A PARADOX CONSIDER A ROLL OF (NON-STICKY) TAPE SITTING ON A FLAT SURFACE. GIVE IT A SMALL SHOVE SO THAT IT STARTS TO UNWIND. AS IT UNWINDS THE C.M. FALLS UNTIL EVENTUALLY IT IS AT THE LEVEL OF THE FLAT SURFACE. WHEN THE TAPE IS FULLY UNWOUND IT LIES AT REST ON THE SURFACE.



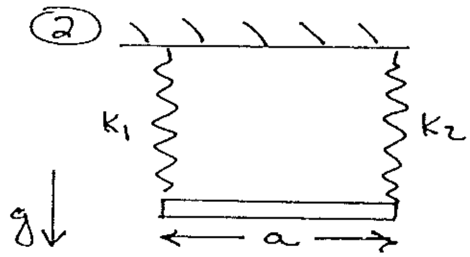
a) WHAT HAS HAPPENED TO THE INITIAL GRAVITATIONAL POTENTIAL ENERGY OF THE C.M.?

b) TO HELP YOUR THINKING, ANALYZE A SIMILAR PROBLEM.

A THIN TAPE OF LENGTH ℓ IS WOUND AROUND A MASSLESS, CYLINDRICAL BOBBIN OF RADIUS r. THE BOBBIN ROLLS DOWN AN INCLINE OF ANGLE α, UNROLLING THE TAPE IN THE PROCESS.



FIND THE VELOCITY OF THE C.M. OF THE BOBBIN AS A FUNCTION OF THE DIMENSIONLESS VARIABLE $u \equiv x/\ell$, WHERE x IS THE DISTANCE THE C.M. OF THE BOBBIN HAS MOVED.

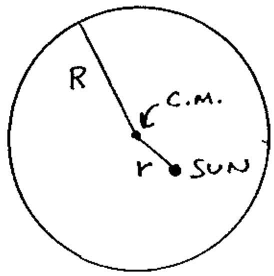
② 

TWO SPRINGS OF EQUAL REST LENGTH SUPPORT A BAR OF MASS m, LENGTH a AS SHOWN.

a) SUPPOSE $k_1 = k_2$ (= SPRING CONSTANTS). WHAT ARE THE NORMAL FREQUENCIES OF OSCILLATION? CONSIDER ONLY MOTION IN A VERTICAL PLANE, AND IN WHICH THE ENDS OF THE BAR MOVE ESSENTIALLY VERTICALLY.

b) SUPPOSE $k_1 \neq k_2$. NOW WHAT ARE THE NORMAL FREQUENCIES? WHAT ARE THE STRETCHES OF THE TWO SPRINGS AT EQUILIBRIUM?

(3) RINGWORLD (COURTESY OF DAN STEVENS)



LARRY NIVEN HAS SUGGESTED THAT A GOOD PLACE TO LIVE WOULD BE ON A LARGE CIRCULAR RING (OR BAND) WHOSE CENTER IS AT THE SUN. THE RING WOULD ROTATE ABOUT ITS C.M. TO PROVIDE AN APPARENT GRAVITY POINTING AWAY FROM THE SUN.

DISCUSS THE STABILITY OF RINGWORLD AGAINST RADIAL DISPLACEMENTS r BETWEEN THE SUN AND THE C.M. OF THE RING. CONSIDER BOTH CASES THE THE C.M. HAS ZERO AND NON-ZERO ANGULAR MOMENTUM ABOUT THE SUN. YOU MAY ASSUME $r \ll R$ WHERE R IS THE RADIUS OF THE RING.

(4) BELOW IS A TRANSLATION OF THE BEGINNING OF A PAPER BY E. SCHRÖDINGER.

3

Quantisation as a Problem of Proper Values (Part I)

(*Annalen der Physik* (4), vol. 79, 1926)

§ 1. IN this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the *node-numbers* of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules.

The usual form of the latter is connected with the Hamilton-Jacobi differential equation,

$$H\left(q, \frac{\partial S}{\partial q}\right) = E. \tag{1}$$

A solution of this equation is sought such as can be represented as the *sum* of functions, each being a function of one only of the independent variables q.

Here we now put for S a new unknown ψ such that it will appear as a *product* of related functions of the single co-ordinates, *i.e.* we put

$$S = K \log \psi. \tag{2}$$

The constant K must be introduced from considerations of dimensions; it has those of *action*. Hence we get

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E. \tag{1'}$$

Now we do *not* look for a solution of equation (1′), but proceed as follows. If we neglect the relativistic variation of mass, equation (1′) can always be transformed so as to become a quadratic form (of ψ and its first derivatives) equated to zero. (For the *one*-electron problem this holds even when mass-variation is not neglected.) We now seek a function ψ, such that for any arbitrary variation of it the integral of the said quadratic form, taken over the whole co-ordinate space,[1] is stationary, ψ being everywhere real, single-valued, finite, and continuously differentiable up to the second order. *The quantum conditions are replaced by this variation problem.*

First, we will take for H the Hamilton function for Keplerian motion, and show that ψ can be so chosen for *all positive*, but only for a *discrete set of negative values of* E. That is, the above variation problem has a discrete and a continuous spectrum of proper values.

The discrete spectrum corresponds to the Balmer terms and the continuous to the energies of the hyperbolic orbits. For numerical agreement K must have the value $h/2\pi$.

The choice of co-ordinates in the formation of the variational equations being arbitrary, let us take rectangular Cartesians. Then (1′) becomes in our case

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0; \tag{1''}$$

e = charge, m = mass of an electron, $r^2 = x^2 + y^2 + z^2$.

Our variation problem then reads

$$\delta J = \delta \iiint dx\,dy\,dz \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2\right] = 0, \tag{3}$$

the integral being taken over all space. From this we find in the usual way . . . .

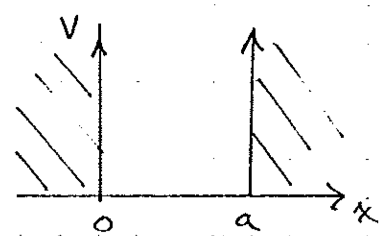
[FOR WHAT ITS WORTH, S = ACTION = $\int L\,dt$, AND $\frac{\partial S}{\partial q} = p$ = GENERALISED MOMENTUM. SEE L & L SEC 43.

THEN YOU CAN GO FROM SCHRODINGER'S (1') TO (1'') BY FOLLOWING HIS SUGGESTION TO REPLACE p BY $\frac{K}{\psi}\frac{\partial \psi}{\partial q}$]

WE CONSIDER A SLIGHT VARIATION ON EQ (1''): ONE DIMENSIONAL MOTION, BUT SUBJECT TO AN ARBITRARY FORCE DERIVED FROM POTENTIAL $V(x)$. THEN

$$(1'') \rightarrow \quad \left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{2M}{K^2}\left(E - V(x)\right)\psi^2$$

WITH THIS FORM OF THE INTEGRAND, CARRY OUT THE VARIATION SUGGESTED IN (3) TO DERIVE SCHRODINGER'S EQUATION.



CONSIDER THE 'INFINITE WELL' POTENTIAL $V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & \text{ELSWHERE} \end{cases}$

SOLVE SCHRODINGER'S EQUATION WITH THIS POTENTIAL, SUPPOSING ψ IS CONTINUOUS AT $x=0$ AND a. CALCULATE THE ALLOWED VALUES OF THE ENERGY E.

THIS SOLUTION IS MEANT TO DESCRIBE THE POSSIBLE MOTION OF A QUANTUM MECHANICAL PARTICLE IN A BOX. THE POSITION OF THE PARTICLE IS UNCERTAIN BY AN AMOUNT $\sim a$. THE MOMENTUM IS KNOWN — BUT NOT ITS SIGN! IN THIS SENSE, THE UNCERTAINTY

IN MOMENTUM IS $\sim p$. WHAT IS THE MINIMUM VALUE OF THE PRODUCT OF THE 'UNCERTAINTIES' pa?

(NOTE: $\psi = 0$ EVERYWHERE IS NOT CONSIDERED AN INTERESTING CASE.)