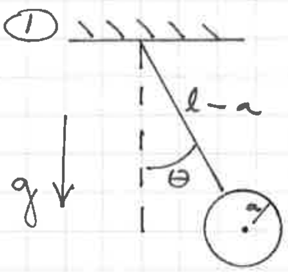
1983

Ph 205 Solutions to Final Exam

(1)



THE PENDULUM IS A SIMPLE PENDULUM NOT A DOUBLE.... THE KEY INSIGHT IS THAT THE ICE ROTATES AS THE PENDULUM SWINGS, WHILE THE WATER DOES NOT. IN THE CASE OF WATER THERE IS NO FRICTION $\Rightarrow$ NO TORQUE ON WATER $\Rightarrow$ NO ROTATION!

$$\left.\begin{aligned} T_{WATER} &= \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \ell^2 \dot{\theta}^2 \\ T_{ICE} &= \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \dot{\theta}^2 = \tfrac{1}{2} m \ell^2 \dot{\theta}^2 + \tfrac{1}{2}\tfrac{2}{5} m a^2 \dot{\theta}^2 \end{aligned}\right\} \equiv \frac{K}{2} m \ell^2 \dot{\theta}^2$$

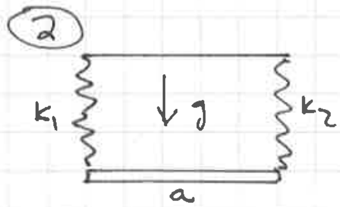
$$V = m g \ell (1 - \cos\theta)$$

SO $\quad K m \ell^2 \ddot{\theta} = - m g \ell \sin\theta$

FOR SMALL θ WE HAVE SIMPLE HARMONIC MOTION WITH $\omega = \sqrt{\dfrac{g}{K\ell}}$

$T = \dfrac{2\pi}{\omega}$ $\quad$ SO $\quad \underline{\dfrac{T_{ICE}}{T_{WATER}} = \sqrt{\dfrac{K_{ICE}}{K_{WATER}}} = \sqrt{1 + \tfrac{2}{5}\, a^2/\ell^2}}$

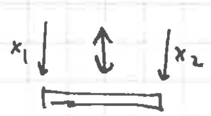
(2)



UNDER THE RESTRICTIONS STATED THERE ARE ONLY 2 DEGREES OF FREEDOM $\Rightarrow$ 2 MODES

a) $k_1 = k_2$. WE USE ELEMENTARY MEANS, TO GET A CHECK ON LAGRANGE'S METHOD IN b)

MODE 1:



$x_1 = x_2 \equiv x$ NO ROTATION

C.M. MOTION: $\quad m\ddot{x} = -(k_1 + k_2)\, x = -2Kx$

$$\boxed{\omega = \sqrt{\frac{2K}{m}}}$$

MODE 2:

$x_1 = -x_2$ $\quad$ CM AT REST ; ROTATION

$$I\ddot{\theta} = N_{ABOUT\ CM} = -2K x_1 \left(\frac{a}{2}\right) = -\frac{1}{2} K a^2 \theta \quad \text{WITH } \theta = \frac{2x_1}{a}$$

$I_{cm} = \frac{1}{12} m a^2$ $\quad$ SO $\quad \ddot{\theta} = -\dfrac{6K}{m}\theta \Rightarrow \boxed{\omega = \sqrt{\dfrac{6K}{m}}}$

b) $K_1 \neq K_2$. AT EQUILIBRIUM WE MUST HAVE $F_1 = F_2$ SO NO TORQUE AND $F_1 + F_2 = Mg$

SO $\quad k_1 x_1 = k_2 x_2 \quad$ AND $\quad k_1 x_1 + k_2 x_2 = Mg \Rightarrow \underline{x_1 = \dfrac{Mg}{2k_1}} \quad \underline{x_2 = \dfrac{Mg}{2k_2}}$

WHERE x_1 AND x_2 ARE MEASURED FROM THE ENDS OF THE UNSTRETCHED SPRINGS.

IF WE SET $q_1 = x_1 - \frac{mg}{2k_1}$ AND $q_2 = x_2 - \frac{mg}{2k_2}$

THEN $V = \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 q_2^2$

AND $T = T_{\text{OF CM}} + T_{\text{ROTATION}} = \frac{1}{2} m \left(\frac{\dot q_1 + \dot q_2}{2}\right)^2 + \frac{1}{2}\cdot\frac{1}{12} m a^2 \left(\frac{\dot q_1 - \dot q_2}{a}\right)^2$ ← NOT $2a$

$$= \frac{1}{2} m \left\{ \frac{1}{3}\dot q_1^2 + \frac{1}{3}\dot q_1 \dot q_2 + \frac{1}{3}\dot q_2^2 \right\}$$

SO THE EQUATIONS OF MOTION ARE

$$\frac{m}{3}\ddot q_1 + \frac{m}{6}\ddot q_2 = -k_1 q_1$$

$$\frac{m}{6}\ddot q_1 + \frac{m}{3}\ddot q_2 = -k_2 q_2$$

TRY $q_1 = A \cos\omega t$ $q_2 = B \cos\omega t$

$$\left(\frac{k_1}{m} - \frac{1}{3}\omega^2\right) A - \frac{\omega^2}{6} B = 0$$

$$-\frac{\omega^2}{6} A + \left(\frac{k_2}{m} - \frac{1}{3}\omega^2\right) B = 0$$

SET DETERMINANT TO ZERO $\Rightarrow \frac{k_1 k_2}{m^2} - \frac{1}{3}\frac{(k_1+k_2)}{m}\omega^2 + \frac{1}{12}\omega^4 = 0$

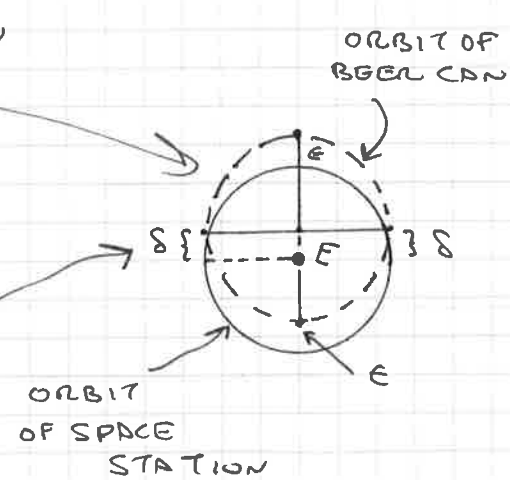
$$\omega^4 - 4\frac{(k_1+k_2)}{m}\omega^2 + 12\frac{k_1 k_2}{m^2} = 0$$

$$\boxed{\omega^2 = +2\frac{(k_1+k_2)}{m} \pm \frac{1}{m}\sqrt{4(k_1+k_2)^2 - 12 k_1 k_2}}$$

CHECK $k_1 = k_2 = k \Rightarrow \omega^2 = \frac{+4k \pm 2k}{m} = \frac{2k}{m}, \frac{6k}{m}$ ✓

(3) BEER CAN IN ORBIT WE CAN SOLVE THIS BY RELATIVELY ELEMENTARY CONSIDERATIONS.

SO LONG AS THE BEER CAN IS NOT THROWN EXTREMELY FAST, IT WILL ORBIT THE EARTH IN AN ELLIPSE (WE IGNORE THE SUN). IF THIS ELLIPSE IS TO APPEAR TO "ORBIT" THE STATION, IT MUST LOOK LIKE

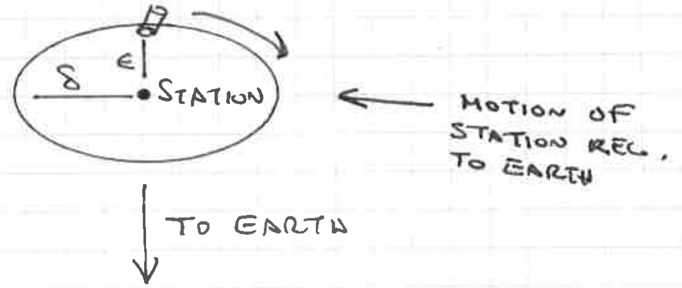


THE PERIOD OF THE ORBIT IS CLEARLY THE SAME AS THAT OF THE SPACE STATION

(KEPLER'S 3RD LAW $T^2 \sim a^2$ WHERE a = SEMI MAJOR AXIS)

NOTE THAT WHEN THE STATION HAS MOVED THRU 90°, THE BEER CAN IS BEHIND THE STATION BY AMOUNT δ.

RELATIVE TO THE STATION THE BEER CAN FOLLOWS A PATH



IF $\epsilon = \delta$ THE PATH IS A CIRCLE.

$\therefore$ THE ASTRONAUT SHOULD THROW THE BEER CAN PARALLEL TO THE ORBIT OF THE STATION ABOUT THE EARTH, AND OPPOSITE TO THE MOTION OF THE STATION.

TO FIND THE VELOCITY, WE NOTE THAT L_{CAN} ABOUT EARTH = SAME AS ANGULAR MOMENTUM OF A CAN ON THE STATION. (WHEN THE STATION HAS MOVED BY 90^0 $r_{CAN} = r_0$, $V_{\theta\, CAN} = V_{\theta}$ STATION

$$\Rightarrow L_{CAN} = M r_0^2 \omega_0 \quad \text{WHERE} \quad \omega_0^2 = \frac{GM_e}{r_0^3}$$

SO $$M(r_0+\epsilon)^2 \omega_i = M r_0^2 \omega_0$$

$$(r_0^2 + 2\epsilon r_0)\omega_i = r_0^2 \omega_0$$

$$\omega_i \sim \omega_0\left(1 - \frac{2\epsilon}{r_0}\right)$$

IF YOU CLAIM THE ASTRONAUT HAS VELOCITY $= (r_0+\epsilon)\omega_0 = v_0\left(1+\frac{\epsilon}{r_0}\right)$ THEN $\Delta V = -2\epsilon \frac{v_0}{r_0} = -2\epsilon\omega_0$

$$V_i = (r_0+\epsilon)\omega_i = r_0\omega_0\left(1+\frac{\epsilon}{r}\right)\left(1-\frac{2\epsilon}{r_0}\right) \sim V_0\left(1-\frac{\epsilon}{r_0}\right)$$

$$\therefore \underline{\Delta V} = -\epsilon\frac{V_0}{r_0} = -\epsilon\omega_0 = \boxed{-\epsilon\sqrt{\frac{GM_e}{r_0^3}}}$$

WE ASSUME THE ASTRONAUT HAS THE SAME VELOCITY AS STATION - OTHERWISE HE DRIFTS AWAY...

FINALLY WE NEED δ. THE ELLIPSE OF THE BEER CAN IS

$$r \doteq r_0 + \epsilon \cos \omega_0 t$$ (RECALL FORM OF SOLUTION TO SMALL OSCILLATION ABOUT CIRCULAR ORBIT)

AND CONSERVATION OF ANGULAR MOMENTUM $\Rightarrow$

$$\dot\theta = \frac{L}{Mr^2} = \frac{L}{Mr_0^2\left(1+\frac{\epsilon}{r_0}\cos\omega_0 t\right)^2} \sim \omega_0\left(1 - \frac{2\epsilon}{r_0}\cos\omega_0 t\right)$$

$$\Rightarrow \theta = \omega_0 t - 2\epsilon\frac{\omega_0}{r_0}\sin\omega_0 t$$

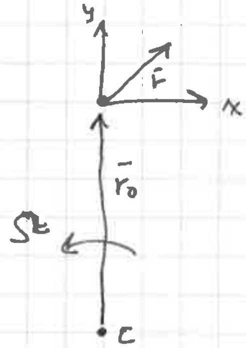
$$\Delta\theta = \theta - \omega_0 t = -2\frac{\epsilon}{r_0}\sin\omega_0 t$$

δ IS MEASURED WHEN $\omega_0 t = 90^0 \Rightarrow \Delta\theta = -\frac{2\epsilon}{r_0}$

AND $\underline{\underline{\delta}} = r_0\Delta\theta = \underline{\underline{-2\epsilon}}$

THE 'ORBIT' IS AN ELLIPSE NOT A CIRCLE

SOLUTION IN THE ROTATING FRAME



IN THE ROTATING FRAME

$$m\bar{a} = \bar{F} - m\bar{\Omega}\times\bar{\Omega}\times(\bar{r}_0+\bar{r}) - 2m\bar{\Omega}\times\bar{v}$$

FOR $\bar{r} \ll \bar{r}_0$ THIS IS APPROXIMATELY

$$m\bar{a} = -\frac{GMm}{(r_0+y)^2}\hat{y} + m\Omega^2(r_0+y)\hat{y} - 2m\Omega\,\hat{z}\times(\dot{x}\hat{x}+\dot{y}\hat{y})$$

$$= \left[-\frac{GMm}{r_0^2}\left(1-\frac{2y}{r_0}\right) + m\Omega^2(r_0+y)\right]\hat{y} + 2m\Omega(\dot{y}\hat{x} - \dot{x}\hat{y})$$

NOW $\quad m\Omega^2 r_0 = \dfrac{GMm}{r_0^2}$

IF INCLUDE z DEPENDENCE OF GRAVITY

SO $\quad m\bar{a} = 3m\Omega^2 y\hat{y} + 2m\Omega(\dot{y}\hat{x} - \dot{x}\hat{y}) \quad - m\Omega^2 z\hat{z}$

WE TRY AN ELLIPSE-LIKE SOLUTION WHICH MIGHT BE APPROPRIATE TO THE INITIAL CONDITION $x(0)=0$, $y(0)=\epsilon$

$x = A\sin\omega t \qquad y = \epsilon\cos\omega t$

x: $\quad \ddot{x} = 2\Omega\dot{y} \Rightarrow -A\omega^2 = -2\Omega\omega\epsilon \Rightarrow A = \dfrac{2\Omega}{\omega}\epsilon$

y: $\quad \ddot{y} = 3\Omega^2 y - 2\Omega\dot{x} \Rightarrow -\epsilon\omega^2 = 3\Omega^2\epsilon - 2\Omega\omega A$

$$= 3\Omega^2\epsilon - 4\Omega^2\epsilon = -\Omega^2\epsilon$$

$\therefore \quad \omega = \Omega \quad$ AND $\quad A = 2\epsilon$

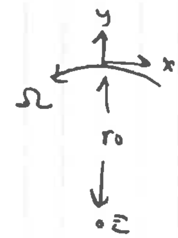
(OR $\omega = -\Omega$ AND $A = -2\epsilon$ WHICH COMES TO THE SAME THING)

THE INITIAL VELOCITY IS $\quad \dot{x}(0) = 2\epsilon\Omega \qquad \dot{y}(0) = 0$

$$= 2\epsilon\sqrt{\frac{GM}{r_0^3}}$$

NOTE THAT $\left.\begin{array}{l}\dot{x}(0) > 0\\ \dot{y}(0) = 0\end{array}\right\} \Rightarrow$ MOTION OPPOSITE TO THAT OF THE STATION, BUT PARALLEL TO THE STATION'S ORBIT

GARBAGE IS THROWN OUT OF A SPACE STATION IN A CIRCULAR ORBIT OF RADIUS r_0 ABOUT THE EARTH. DESCRIBE THE MOTION OF THE GARBAGE SUPPOSING IT IS GIVEN SMALL INITIAL VELOCITY V_x, V_y OR V_z RELATIVE TO THE SPACE STATION



QUICK SOLUTIONS!

a) $\Delta V = V_z$. L_z' = SAME AS L_z FOR MOTION IN A PLANE TILTED BY

$\Theta = \frac{V_z}{\Omega r_0}$. E REMAINS SAME TO 1ST ORDER IN V_z

($KE \to KE + \frac{1}{2} m V_z^2$; PE = SAME)

∴ ORBIT REMAINS A CIRCLE, BUT TILTED BY Θ

b) $\Delta V = V_y$. ORBIT IS AN ELLIPSE IN THE ORIGINAL PLANE

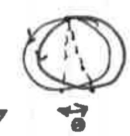
AGAIN KE ≈ SAME, PE ≈ SAME ⇒ E = SAME

ALSO L = SAME

∴ MAJOR AXIS OF ELLIPSE REMAINS $2r_0$

FOR SMALL ECCENTRICITY, ELLIPSE ~ CIRCLE

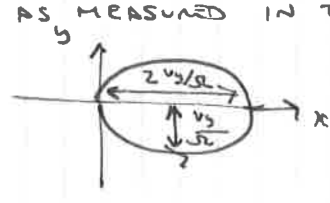
∴ NEW ORBIT SAME AS ORIGINAL BUT ROTATED BY $\Theta = \frac{V_y}{\Omega r_0}$



Δ PERIGEE $= r_0 \Theta = \frac{V_y}{\Omega}$

FOR $V_y < 0$

THE MOTION RELATIVE TO THE STATION, AS MEASURED IN THE ROTATING x-y FRAME IS (FOR $V_y < 0$)



c) $\Delta V = V_x < 0$. NOW $KE > KE_i$, $L_z > L_i$ ⇒ ORBIT IS BIGGER & HAS LONGER PERIOD.

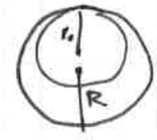
PE = SAME

E > E_i

FOR $V_x < 0$ THE ORIGINAL PT IS THE PERIGEE

(FOR $V_x > 0$, IT WOULD BE APOGEE.

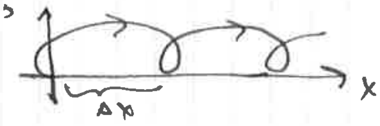
FOR SMALL ECCENTRICITY THE ORBIT IS STILL A CIRCLE.



NOW $m \frac{(V_i + V_x)^2}{R} = \frac{GM}{r_0^2} = \frac{V_i^2}{r_0} \Rightarrow R \approx r_0 \left(1 + 2\frac{V_x}{V_i}\right)$

APOGEE IS $2R - r_0 = r_0 \left(1 + 4\frac{V_x}{V_i}\right)$ $\quad V_i = r_0 \Omega$.

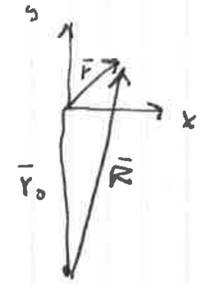
IN THE ROTATING FRAME ORBIT LOOKS LIKE



$\Delta x \approx V_i \Delta T \approx V_i T_i \left[\left(\frac{R}{r_0}\right)^{3/2} - 1\right] = V_i T_i \cdot 3\frac{V_x}{V_i} = 3 V_x T_i = \frac{6\pi V_i}{\Omega}$

2

SOLUTION IN THE ROTATING FRAME



$$\bar{a} = \frac{\bar{F}}{m} - \bar{\Omega} \times (\bar{\Omega} \times \bar{R}) - 2\bar{\Omega} \times \bar{v}$$

$$R = x\hat{x} + (r_0 + y)\hat{y} + z\hat{z}$$

$$\bar{\Omega} = \Omega\hat{z} \qquad \bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$\frac{\bar{F}}{m} = -\frac{GM}{R^3}\bar{R} = -\frac{GM}{R^2}\left(\frac{x\hat{x}}{R} + \frac{r_0+y}{R}\hat{y} + \frac{z}{R}\hat{z}\right)$$

$$\approx -\frac{GM}{r_0^3}x\hat{x} - \frac{GM\hat{y}}{(r_0+y)^2} - \frac{GMz}{r_0^3}\hat{z} \qquad \frac{GM}{r_0^3} = \Omega^2$$

$$\sim -\Omega^2 x\hat{x} - \Omega^2 r_0\left(1 - \frac{2y}{r_0}\right)\hat{y} - \Omega^2 z\hat{z}$$

$$\bar{\Omega} \times \bar{R} = -\Omega(r_0+y)\hat{x} + \Omega x\hat{y} \qquad \bar{\Omega} \times (\bar{\Omega} \times \bar{R}) = -\Omega^2(r_0+y)\hat{y} - \Omega^2 x\hat{x}$$

$$\bar{\Omega} \times \bar{v} = \Omega\dot{x}\hat{y} - \Omega\dot{y}\hat{x}$$

so $\bar{a} = -\Omega^2 x\hat{x} - \Omega^2 r_0\hat{y} + 2\Omega^2 y\hat{y} - \Omega^2 z\hat{z} + \Omega^2(r_0+y)\hat{y} + \Omega^2 x\hat{x} + 2\Omega\dot{y}\hat{x} - 2\Omega\dot{x}\hat{y}$

$$\bar{a} = 2\Omega\dot{y}\hat{x} + (3\Omega^2 y - 2\Omega\dot{x})\hat{y} - \Omega^2 z\hat{z}$$

$$\ddot{x} = 2\Omega\dot{y}$$

$$\ddot{y} = 3\Omega^2 y - 2\Omega\dot{x}$$

$$\ddot{z} = -\Omega^2 z$$

INITIAL CONDITIONS: $\bar{r} = 0 \qquad \dot{\bar{r}} = (v_{0x}, v_{0y}, v_z)$

AT ONCE $\quad z = \frac{v_z}{\Omega}\sin\Omega t$

FOR x & y WE TRY OSCILLATORY SOLUTIONS – WITH A POSSIBLE OFFSET. IN ADDITION, THE SOLUTION IN x MAY DRIFT ....

$$x = A + B\cos\omega t + C\sin\omega t + Dt$$

$$y = E + F\cos\omega t + G\sin\omega t$$

$$x(0) = 0 = A + B \qquad y(0) = 0 = E + F$$

$$x = A(1-\cos\omega t) + C\sin\omega t + Dt$$

$$y = E(1-\cos\omega t) + G\sin\omega t$$

$$\ddot{x} = 2\Omega\dot{y} \Rightarrow A\omega^2\cos\omega t - C\omega^2\sin\omega t = 2E\Omega\omega\sin\omega t + 2G\omega\Omega\cos\omega t$$

so $\quad A\omega = 2G\Omega \quad$ AND $\quad C\omega = -2E\Omega$

$\ddot{y} = 3\Omega^2 y - 2\Omega\dot{x} \Rightarrow E\omega^2\cos\omega t - G\omega^2\sin\omega t = 3\Omega^2 E(1-\cos\omega t) + 3\Omega^2 G\sin\omega t$
$- 2\Omega\omega A\sin\omega t - 2\Omega\omega C\cos\omega t + 2D\Omega$

so $E\omega^2 = -3E\Omega^2 + 4E\Omega^2 = E\Omega^2 \Rightarrow \omega = \Omega$

$-G\omega^2 = 3G\Omega^2 - 4G\Omega^2 \Rightarrow \omega = \Omega$

and $3\Omega^2 E - 2D\Omega = 0 \Rightarrow D = \frac{3}{2}E\Omega$

$x = 2G(1-\cos\Omega t) - 2E\sin\Omega t + \frac{3}{2}E\Omega t$

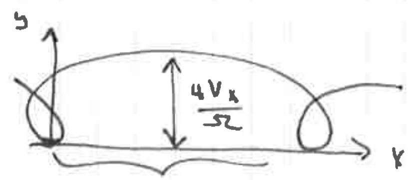
$y = E(1-\cos\Omega t) + G\sin\Omega t$

$\dot{x}(0) = -2E\Omega + \frac{3}{2}E\Omega = -\frac{1}{2}E\Omega = v_x \Rightarrow E = -\frac{2v_x}{\Omega}$

$\dot{y}(0) = G\Omega = v_y \Rightarrow G = \frac{v_y}{\Omega}$

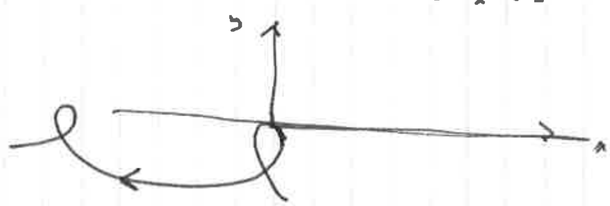
$$\boxed{\begin{aligned} x &= \frac{2v_y}{\Omega}(1-\cos\Omega t) + 4\frac{v_x}{\Omega}\sin\Omega t - 3v_x t \\ y &= -\frac{2v_x}{\Omega}(1-\cos\Omega t) + \frac{v_y}{\Omega}\sin v_y t \\ z &= \frac{v_z}{\Omega}\sin\omega t \end{aligned}}$$

$v_x < 0,\ v_y = 0$

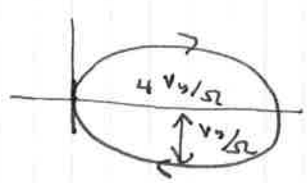


$\Delta x = 3v_x T = 6\pi v_x/\Omega$

$v_x > 0,\ v_y = 0$

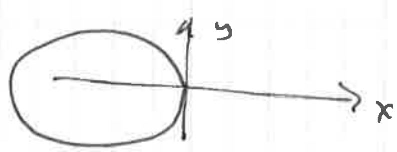


$v_x = 0,\ v_y > 0$

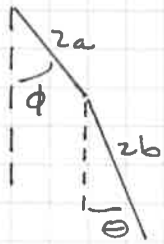


SPACE STATION MOVES IN THE $-X$ DIRECTION.

$v_x = 0,\ v_y < 0$



(4) a) WE USE LAGRANGE TO FIND THE EQUATION OF MOTION OF θ.



THIS IS A BIT OF A GRIND:

$$V = mga(1-\cos\phi) + mg\left[2a(1-\cos\phi) + b(1-\cos\theta)\right] + \text{CONST}$$

$$= mgb(1-\cos\theta) + f(\phi)$$

$$T_a = \tfrac{1}{2} m a^2 \dot\phi^2 + \tfrac{1}{2}\cdot\tfrac{1}{12}(2a)^2\dot\phi^2 = \tfrac{1}{2}\cdot\tfrac{4}{3} m a^2 \dot\phi^2$$

$$T_b = \tfrac{1}{2} m v_{cm}^2 + \tfrac{1}{2}\cdot\tfrac{1}{3} m b^2 \dot\theta^2$$

$x_b = 2a\sin\phi + b\sin\theta \qquad \dot x_b = 2a\cos\phi\,\dot\phi + b\cos\theta\,\dot\theta$

$y_b = 2a\cos\phi + b \quad \theta \qquad \dot y_b = -2a\sin\phi\,\dot\phi - b\sin\theta\,\dot\theta$

$$v_b^2 = 4a^2\dot\phi^2 + 4ab\cos(\theta-\phi)\dot\phi\dot\theta + b^2\dot\theta^2$$

so $L = \tfrac{1}{2}\tfrac{4}{3} m b^2\dot\theta^2 + 2mab\cos(\theta-\phi)\dot\theta\dot\phi - mgb(1-\cos\theta) + F(\phi)$

$$\frac{\partial L}{\partial\dot\theta} = \tfrac{4}{3} m b^2\dot\theta + 2mab\cos(\theta-\phi)\dot\phi$$

$$\boxed{\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} &= \tfrac{4}{3} m b^2\ddot\theta + 2mab\left(\cos(\theta-\phi)\ddot\phi - \sin(\theta-\phi)(\dot\theta-\dot\phi)\dot\phi\right. \\ &= \frac{\partial L}{\partial\theta} = -mgb\sin\theta - 2mab\sin(\theta-\phi)\dot\theta\dot\phi\end{aligned}}$$

WHAT A MESS!

b) $t_{\mathrm{I}\to\mathrm{II}} = t_{\mathrm{III}\to\mathrm{IV}} = \dfrac{\pi}{\omega}$

$t_{\mathrm{II}\to\mathrm{III}} \sim t_{\mathrm{IV}\to\mathrm{V}} \approx \dfrac{\pi}{\Omega}$ WHERE Ω = ANGULAR VELOCITY OF PENDULUM MOTION OF ROD 2b

FROM ABOVE, WITH $\phi = \dot\phi = \ddot\phi = 0$ $\quad \tfrac{4}{3} m b^2\ddot\theta \sim -mgb\theta$

$$\Rightarrow \Omega = \sqrt{\frac{3g}{4b}}$$

$$T_{\mathrm{I}\to\mathrm{V}} = \frac{2\pi}{\omega} + 2\pi\sqrt{\frac{4b}{3g}}$$

WHAT IS ω?

THE KINETIC ENERGY OF ROD b AT TIME II MUST BE SUFFICIENT FOR THE ROD TO RISE TO THE VERTICAL

$$\Rightarrow KE \geq 2mgb$$

Ph 205 FINAL EXAM \5

IF $a = \frac{b}{2}$, ROD b IS ROTATING ABOUT ITS CM WHICH IS AT REST FROM t_I TO t_{II}

$\therefore$ $KE = \frac{1}{2} I_{cm}\omega^2 = \frac{1}{6} m b^2 \omega^2$ JUST BEFORE t_{II}

(IT'S CM WAS NOT FALLEN AT ALL $\Rightarrow$ NO INCREASE IN K.E. DUE TO GRAVITY!)

A QUICK USE OF THIS RESULT SUGGESTS SETTING

$$\frac{1}{6} m b^2 \omega^2 = 2mgb \Rightarrow \omega^2 = \frac{12g}{b}$$

AND SO $T_{I \to V} = \frac{2\pi}{2}\sqrt{\frac{b}{3g}} + 4\pi\sqrt{\frac{b}{3g}} = 5\pi\sqrt{\frac{b}{3g}} \sim .6\text{ SEC}$

FOR $b = 5$ cm

HOWEVER, AT MOMENT t_{II} THE UPPER END OF ROD b BECOMES SUDDENLY FIXED $\Rightarrow$ IMPULSE WHICH EATS A LOT OF K.E.

DURING THE IMPULSE L ABOUT JOINT OF a TO b = CONST

$$L_{JOINT} = L_{OF\ CM\ MOTION} + I_{cm}\,\omega = I_{cm}\,\omega = \frac{1}{3} m b^2 \omega$$

AFTER THE IMPULSE $L_{JOINT} = I_{JOINT}\,\omega' = \frac{4}{3} m b^2 \omega'$

SO $\omega' = \frac{1}{4}\omega \Rightarrow KE_{AFTER} = \frac{1}{2} I_{JOINT}\,\omega'^2 = \frac{1}{2}\cdot\frac{4}{3} m b^2 \frac{\omega^2}{16}$

$$= \frac{1}{24} m b^2 \omega^2 = 2mgb$$

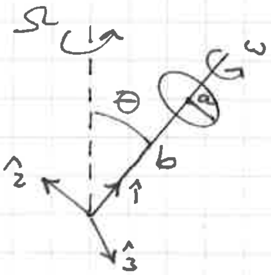
TO RISE TO TOP

SO $\omega^2 = 48g/b$

AND $T_{I-V} = \frac{2\pi}{4}\sqrt{\frac{b}{3g}} + 4\pi\sqrt{\frac{b}{3g}} = \frac{9\pi}{2}\sqrt{\frac{b}{3g}} \sim .5\text{ SEC}$

(5)

MOST PEOPLE PROCEDED AS FOLLOWS:



SET UP A LOCAL COORD SYSTEM $\hat{1}, \hat{2}, \hat{3}$ WHICH ROTATES ABOUT THE VERTICAL AT RATE $\bar{\Omega} = (\Omega\cos\theta, \Omega\sin\theta, 0)$

CALCULATE TORQUES (AND MOMENTS OF INERTIA) ABOUT THE PIVOT POINT TO AVOID THE HORIZONTAL FORCE ON THE PIVOT ($= M\Omega^2 b \sin\theta$ FOR WHAT IT'S WORTH)

THEN $I_1 = ma^2$, $I_2 = I_3 = m\left(b^2 + \frac{a^2}{2}\right)$ BY PARALLEL AXIS THM.

THE METHOD OF ANALYSIS IS A VARIATION ON EULER'S EQUATIONS:

$\bar{N} = \frac{d\bar{L}}{dt} = \frac{\delta L}{\delta t} + \bar{\Omega} \times \bar{L}$ WHERE $\frac{\delta \bar{L}}{\delta t} = \bar{\bar{I}} \cdot \dot{\bar{\omega}}_{TOT}$ IN THE ROTATING FRAME. NOTE THAT WE USE $\bar{\Omega} \times \bar{L}$ NOT $\bar{\omega}_{TOT} \times \bar{L}$

$$\bar{N} = b\hat{1} \times (-mg\hat{z}) = -mgb\sin\theta\,\hat{3}$$

SO $N_3 = \underbrace{I_3 \dot{\omega}_3}_{0} + \Omega_1 I_2 \omega_2 - \Omega_2 I_1 \omega_1$

THERE IS AN AMBIGUITY OF THE WORDING OF THE PROBLEM AS TO HOW TO DESCRIBE $\bar{\omega}_{TOT}$. I HAD INTENDED

$\bar{\omega}_{TOT} = (\omega + \Omega\cos\theta, \Omega\sin\theta, 0)$ BUT $\bar{\omega}_{TOT} = (\omega, \Omega\sin\theta, 0)$ WILL BE ACCEPTED ALSO. HERE I USE THE FIRST DEFINITION

SO $-mgb\sin\theta = \Omega\cos\theta\, I_2 \Omega\sin\theta - \Omega\sin\theta\, I_1(\omega + \Omega\cos\theta)$

$$(I_2 - I_1)\Omega^2\cos\theta\sin\theta - I_1\omega\Omega\sin\theta + mgb\sin\theta = 0$$

$$\boxed{\Omega = \frac{I_1\omega \pm \sqrt{I_1^2\omega^2 - 4mgb(I_2 - I_1)\cos\theta}}{2(I_2 - I_1)\cos\theta}}$$

[IF YOU USED THE DEFINITION $\omega_1 = \omega$, THEN $(I_2 - I_1) \rightarrow I_2$ ABOVE]

FOR STEADY MOTION WE NEED $\omega > \frac{2}{I_1}\sqrt{mgb(I_2 - I_1)\cos\theta}$

$$\boxed{\omega > \frac{2}{a^2}\sqrt{gb(b^2 - a^2/2)\cos\theta}}$$

FOR LARGE ω, $\Omega \rightarrow I_1\omega\frac{(1 \pm (1 - 2mgb(I_2 - I_1)\cos\theta)/I_1^2\omega^2)}{2(I_2 - I_1)\cos\theta}$ $\overset{-\text{ ROOT}}{=}$ $\boxed{\frac{mgb}{I_1\omega} = \frac{gb}{a^2\omega}}$

I ACTUALLY HAD IN MIND A MORE DEVIOUS APPROACH:

IN THE ROTATING FRAME THE C.M. IS AT REST AND THE AXLE IS AT REST, ALTHOUGH THE HOOP SPINS ABOUT THE AXLE. THIS IS A KIND OF STATIC EQUILIBRIUM.

$$\therefore \quad \sum \bar{F} = 0 \quad \text{AND} \quad \sum \bar{N} = 0 \quad \text{MUST HOLD IN THIS FRAME.}$$

THE KEY IS $\sum \bar{N} = 0$. THERE ARE 3 KINDS OF TORQUES

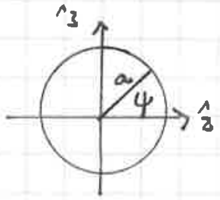
$$\bar{N}_{GRAVITY} = -mgb\sin\theta\,\hat{3}$$

(ABOUT THE PIVOT. WE USE THE SAME COORD SYSTEM AS ON p 6)

$$\bar{N}_{CENTRIFUGAL} = -\int dm\ \bar{r}\times(\bar{\Omega}\times(\bar{\Omega}\times\bar{r}))$$

$$\bar{N}_{CORIOLIS} = -2\int dm\ \bar{r}\times(\bar{\Omega}\times\dot{\bar{r}}) = -2\int dm\ \bar{r}\times(\bar{\Omega}\times(\bar{\omega}\times\bar{r}))$$

TO INTEGRATE AROUND THE HOOP, DEFINE ANGLE ψ IN THE $\hat{2}-\hat{3}$ PLANE



SO $dm = \frac{m}{2\pi}d\psi$

THEN $\bar{r} = (b,\ a\cos\psi,\ a\sin\psi)$

$\bar{\Omega} = (\Omega\cos\theta,\ \Omega\sin\theta,\ 0)$

$\bar{\omega} = (\omega,\ 0,\ 0)$

CENTRIFUGAL PIECE: $\bar{r}\times(\bar{\Omega}\times(\bar{\Omega}\times\bar{r})) = \bar{r}\times[(\bar{\Omega}\cdot\bar{r})\bar{\Omega} - \Omega^2\bar{r}] = (\bar{\Omega}\cdot\bar{r})(\bar{r}\times\bar{\Omega})$

$\bar{\Omega}\cdot\bar{r} = b\Omega\cos\theta + a\Omega\sin\theta\cos\psi$

$$\bar{r}\times\bar{\Omega} = \begin{vmatrix} 1 & 2 & 3 \\ b & a\cos\psi & a\sin\psi \\ \Omega\cos\theta & \Omega\sin\theta & 0 \end{vmatrix} = -a\Omega\sin\theta\sin\psi\,\hat{1} + a\Omega\cos\theta\sin\psi\,\hat{2} + (b\Omega\sin\theta - a\Omega\cos\theta\cos\psi)\,\hat{3}$$

SO $\bar{N}_{CENT} = -m\{b^2\Omega^2\cos\theta\sin\theta - \frac{1}{2}a^2\Omega^2\sin\theta\cos\theta\}\hat{3}$

NOTING $\int d\psi\sin\psi = \int d\psi\cos\psi = \int d\psi\sin\psi\cos\psi = 0$ ETC

CORIOLIS PIECE: $\bar{r}\times(\bar{\Omega}\times(\bar{\omega}\times\bar{r})) = \bar{r}\times[(\bar{\Omega}\cdot\bar{r})\bar{\omega} - (\bar{\Omega}\cdot\bar{\omega})\bar{r}] = (\bar{\Omega}\cdot\bar{r})\,\bar{r}\times\bar{\omega}$

$$\bar{r}\times\bar{\omega} = \begin{vmatrix} 1 & 2 & 3 \\ b & a\cos\psi & a\sin\psi \\ \omega & 0 & 0 \end{vmatrix} = -a\omega\cos\psi\,\hat{3}$$

SO $\bar{N}_{CORIOLIS} = +\frac{2M}{2}a^2\omega\Omega\sin\theta\,\hat{3}$

SO $\sum\bar{N} = 0 = M(b^2 - \frac{a^2}{2})\Omega^2\cos\theta\sin\theta - Ma^2\omega\Omega\sin\theta + mgb\sin\theta$

AS BEFORE...

PH 205 FINAL EXAM

18

(6) MY EQUATION (1'') HAD 2 ERRORS, WHICH SOME OF YOU NOTED BY COMPARING WITH SCHRODINGERS (1'') AND (3)

$$(1'') \rightarrow f(\psi,\psi') = \left(\frac{\partial\psi}{\partial x}\right)^2 - \frac{2m}{k^2}\left(E - V(x)\right)\psi^2$$

WE WANT $\delta I = \delta\int f(\psi,\psi')\,dx = 0$

THIS IS THE STANDARD EULER-LAGRANGE PROBLEM, WHOSE SOLUTION OBEYS LAGRANGE'S EQUATION:

$$\frac{d}{dx}\frac{\partial f}{\partial\psi'} - \frac{\partial f}{\partial\psi} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{k^2}(E-V)\psi = 0}$$

IF YOU USED MY ORIGINAL FORM OF (1''), YOU WOULD GET

$$\frac{\partial^2\psi}{\partial x^2} + \frac{m}{k^2}(E+V) = 0$$

WHICH IS NOT VERY SOLUABLE FOR V GIVEN AS BELOW

FOR $V = \begin{cases} 0 & 0 < x < a \\ \infty & \text{ELSEWHERE} \end{cases}$

WE SEE THAT OUTSIDE $(0,a)$, ψ MUST VANISH FOR FINITE E, SINCE $V \to \infty$ THERE.

ON $(0,a)$ WE HAVE $\psi'' + \frac{2mE}{k^2}\psi = 0$

$$\Rightarrow \psi = A\cos\sqrt{\frac{2mE}{k^2}}\,x + B\sin\sqrt{\frac{2mE}{k^2}}\,x$$

TO BE CONTINUOUS AT $x=0$ AND a, ψ MUST VANISH THERE

$$\Rightarrow A = 0, \quad \sqrt{\frac{2mE}{k^2}} = \frac{n\pi}{a}$$

OR $\boxed{E = \frac{k^2}{2m}\left(\frac{n\pi}{a}\right)^2}$

IF YOU BELIEVE $E = \frac{p^2}{2m}$ STILL HOLDS, THEN $P = \frac{n\pi k}{a}$

AND $Pa = n\pi k$ SO $\boxed{Pa_{MIN} = \pi k = \frac{h}{2}}$

IF YOU FOLLOW SCHRODINGER: $P = \frac{k}{\psi}\frac{\partial\psi}{\partial x} = \frac{n\pi k}{a}\cot\sqrt{\frac{2mE}{k^2}}\,x \approx \frac{n\pi k}{a}$