Some Properties of Sourceless Wave Packets

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We consider a 3-dimensional wave function $f(\mathbf{x}, t)$ that obeys the linear, homogeneous (source-free) wave equation,

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}, \tag{1}$$

where c is the wave speed. The wave equation (1) supports monochromatic plane-wave solutions of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ where the wave vector $\mathbf{k}$ is related to the angular frequency $\omega = kc$ via the wave number $k = |\mathbf{k}|$. Then, the linearity of the wave equation permits us to synthesize an arbitrary wave $f(\mathbf{x}, t)$ out of plane waves,

$$f(\mathbf{x}, t) = \int F(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k}, \tag{2}$$

in which case the Fourier amplitudes are given by the inverse form,

$$F(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int f(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{x}. \tag{3}$$

The wave equation (1) places a constraint on the Fourier amplitudes $F(\mathbf{k}, t)$, which we learn by inserting the expansion (2) into eq. (1). Thus,

$$\int \left(k^2 F + \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} \right) e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k} = 0. \tag{4}$$

This equation must be true for all $\mathbf{x}$, so the quantity in parentheses (which does not depend on $\mathbf{x}$) must vanish. Therefore,

$$F(\mathbf{k}, t) = A e^{i\omega t} + B e^{-i\omega t}, \qquad \frac{\partial F(\mathbf{k}, t)}{\partial t} = i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}, \tag{5}$$

so,

$$F(\mathbf{k}, 0) = A + B, \qquad \frac{\partial F(\mathbf{k}, 0)}{\partial t} = i\omega(A - B), \tag{6}$$

$$A = \frac{1}{2} \left(F(\mathbf{k}, 0) - \frac{i}{\omega} \frac{\partial F(\mathbf{k}, 0)}{\partial t} \right), \qquad B = \frac{1}{2} \left(F(\mathbf{k}, 0) + \frac{i}{\omega} \frac{\partial F(\mathbf{k}, 0)}{\partial t} \right). \tag{7}$$

The Fourier expansion (2) can now be written as,

$$f(\mathbf{x}, t) = \frac{1}{2} \int [F(\mathbf{k}) + i\dot{F}(\mathbf{k})/\omega] e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, d^3\mathbf{k} + \frac{1}{2} \int [F(\mathbf{k}) - i\dot{F}(\mathbf{k})/\omega] e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}\, d^3\mathbf{k}, \tag{8}$$

where,

$$F(\mathbf{k}) = F(\mathbf{k}, 0) = \frac{1}{(2\pi)^3} \int f(\mathbf{x}, 0) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{x}, \tag{9}$$

and,

$$\dot{F}(\mathbf{k}) = \frac{\partial F(\mathbf{k},0)}{\partial t} = \frac{1}{(2\pi)^3} \int \frac{\partial f(\mathbf{x},0)}{\partial t} e^{-i\mathbf{k}\cdot\mathbf{x}}\ d^3\mathbf{x}. \tag{10}$$

A consequence of expression (8) is that if the wave function $f(\mathbf{x},t)$ vanishes for all $\mathbf{x}$ for any finite time interval, we can redefine $t = 0$ to be the center of that interval, such that all Fourier amplitudes $F(\mathbf{k})$ as well as $\dot{F}(\mathbf{k})$ vanish, and we learn that the wave function was actually zero for all time.

Summary: **Nontrivial solutions to the 3-dimensional homogeneous wave equation cannot vanish for all x over any finite time interval.** This result actually holds for any number of spatial dimensions.

The spacetime converse of this result also holds. That is, **the wavefunction cannot vanish for all time over any spatial region of finite extent**.

In one dimension, coordinates x and t are formally interchangeable, so in one dimension it follows from the preceding analysis that the wavefunction cannot vanish for all time over any spatial region of finite extent. Of course, standing waves such as $f(x,t) = \sin(kx)\sin(\omega t)$ vanish at all times for the set of measure zero $\{x = n\pi/k\}$, and vanish at all x for the set $\{t = n\pi/\omega\}$.

When there are more than one spatial dimension, we could apply scalar diffraction theory to a bounding surface entirely within the spatial region where the wavefunction vanishes for all times. Then, the wavefunction vanishes at any other point according to the diffraction integral.

Appendix: Can a (nontrivial) wavefunction be zero outside a bounded region at a given time?

The answer is YES, but only if the synthesis (2) of the wavefunction invloves an unbounded range of wave vectors $\mathbf{k}$.

We extrapolate from a standard argument of Fourier analysis.[1] Suppose that at time t the wavefunction (2) is zero for all $|\mathbf{x}| \geq a$. Without loss of generality, we can set $t = 0$. Suppose also that the Fourier amplitudes $F(\mathbf{k})$ are zero for all $|\mathbf{k}| \geq b$. Then,

$$f(\mathbf{x},0) = \int_{|\mathbf{k}|<b} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\ d^3\mathbf{k} = 0, \qquad \text{for } |\mathbf{x}| \geq a. \tag{11}$$

In particular, the wavefunction and its derviatives vanish at $\mathbf{a} = a(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$, so we also have,

$$\frac{\partial^{l+m+n} f(\mathbf{a},0)}{\partial x^l \partial y^m \partial z^n} = i^{l+m+n} \int_{|\mathbf{k}|<b} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{a}} k_x^l k_y^m k_z^n\ d^3\mathbf{k} = 0. \tag{12}$$

Thus, for any $\mathbf{x}$,

$$f(\mathbf{x},0) \;=\; \int_{|\mathbf{k}|<b} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\ d^3\mathbf{k} = \int_{|\mathbf{k}|<b} F(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{a})} e^{i\mathbf{k}\cdot\mathbf{a}}\ d^3\mathbf{k}$$

[1] See, for example, sec. 2.9 of H. Dym and H.P. McKean, *Fourier Series and Integrals* (Academic Press, 1972).

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} \frac{[i(x-a)]^l}{l!} \sum_{m=0}^{\infty} \frac{[i(y-a)]^m}{m!} \sum_{n=0}^{\infty} \frac{[i(z-a)]^m}{n!} \int_{|\mathbf{k}|<b} F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{a}} k_x^l k_y^m k_z^n \; d^3\mathbf{k} \\
&= 0. \qquad (13)
\end{aligned}
$$

This contradicts our hypothesis, and we conclude that a wavefunction cannot be both bounded in ordinary space and have a Fourier analysis with wave vectors that are bounded in $\mathbf{k}$-space.