A Simple 3-Body Gravitational Problem

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1 Problem

In general, there is no analytic solution for the motion of a 3-body gravitational problem. Here is a simple version that permits an analytic solution.

Three equal masses m are initially at rest along the x-axis with separations R. At time $t = 0$ the "outer" two masses become free to move, towards the center mass that remains at rest at $x = 0$.

2 Solution

For $t > 0$ the equation of motion of each of the "outer" accelerate according to Newton's law of gravitation with acceleration of magnitude a related by,

$$ma = \frac{Gm^2}{r^2} + \frac{Gm^2}{4r^2}\,, \qquad a = \frac{dv}{dt} = \frac{5Gm}{4r^2}\,, \tag{1}$$

where G is Newton's gravitational constant, r is the distance between each of the two "outer" masses and the "center" mass, and v is the magnitude of the velocity of each of the "outer" masses.

A first integral of eq. (1) is obtained from conservation of energy U,

$$\begin{aligned} U &= \mathrm{KE} + \mathrm{PE} = U_0 = -2\frac{Gm^2}{R} - \frac{Gm^2}{2R} = -\frac{5Gm^2}{2R} \\ &= U(r) = 2\frac{mv^2}{2} - 2\frac{Gm^2}{r} - \frac{Gm^2}{2r} = mv^2 - \frac{5Gm^2}{2r}\,. \end{aligned} \tag{2}$$

Hence,

$$v = \frac{dr}{dt} = \sqrt{\frac{5Gm}{2}}\sqrt{\frac{1}{r} - \frac{1}{R}}, \qquad dr\sqrt{\frac{r}{R-r}} = \sqrt{\frac{5Gm}{2R}}dt \tag{3}$$

With $x = R - r$, x goes from R to 0 as time increases from 0 to t when the masses collide. Using 195.01 and 195.04 of [1] with $a = 0$, $b = 1$, $f = R$, $g = -1$ and $k = ag - bf = -R$,

$$\begin{aligned} \sqrt{\frac{5Gm}{2R}}t &= -\int_R^0 dx\,\sqrt{\frac{R-x}{x}} = \int_0^R dx\,\sqrt{\frac{R-x}{x}} = \sqrt{x}\sqrt{R-x}\Big|_0^R + \frac{R}{2}\int_0^R \frac{dx}{\sqrt{x}\sqrt{R-x}} \\ &= -\frac{R}{2}\sin^{-1}\frac{R-x}{R}\Big|_0^R = \frac{\pi R}{4}\,, \end{aligned} \tag{4}$$

so the collision occurs at time,

$$t = \frac{\pi R}{4}\sqrt{\frac{2R}{5Gm}}\,. \tag{5}$$

A Appendix: Another Simple Example

The present example is closely related to the first considerations of the gravitational 3-body problem by Euler in 1767 [2, 3]. Another simple example is the case that two masses m are somehow held fixed at $x = \pm R$ along the x-axis, and mass M is near the origin. The equilibrium point for mass M is the origin, and this equilibrium is stable against small perturbations.

The gravitational potential in the x-y planes for mass M is,

$$\begin{aligned} U(x,y) &= -\frac{GMm}{\sqrt{(R+x)^2+y^2}} - \frac{GMm}{\sqrt{(R+x)^2+y^2}} \\ &\approx U(0,0) + \frac{1}{2}\frac{\partial^2 U(0,0)}{\partial x^2}x^2 + \frac{1}{2}\frac{\partial^2 U(0,0)}{\partial y^2}y^2 + \frac{\partial^2 U(0,0)}{\partial x \partial y}xy. \end{aligned} \tag{6}$$

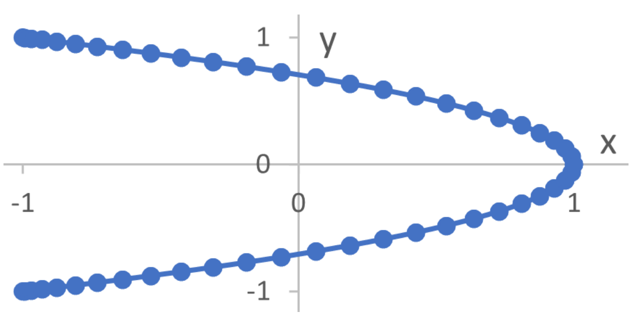
The terms in the potential in x^2 and y^2 imply that the angular frequencies for small oscillations of M about the origin are,

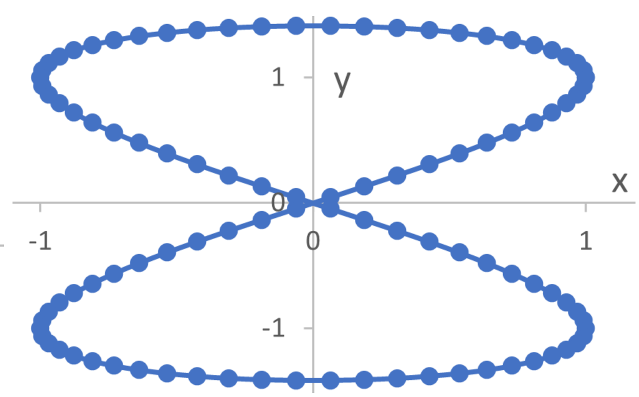
$$\omega_x = \sqrt{\frac{\partial^2 U(0,0)/\partial x^2}{M}} = \sqrt{\frac{8Gm}{R^3}}\,, \qquad \omega_y = \sqrt{\frac{\partial^2 U(0,0)/\partial y^2}{M}} = \sqrt{\frac{2Gm}{R^3}} = \frac{\omega_x}{2}\,. \tag{7}$$

For small oscillations in z, $\omega_z = \omega_y$.

Note that if mass M were in a circular orbit of radius R about fixed mass m, its angular velocity would be $\Omega = \sqrt{Gm/R^3}$ (Kepler's 3rd law).

The motion for small oscillations of mass M of the form $x = \cos(\omega_x t)$ and $y = \sin(\omega_y t)$ is sketched on the left below, while the right figure is for $x = \cos(\omega_x t)$ and $y = \cos(\omega_y t) + \sin(\omega_y t)$.





References

[1] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

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[3] S.R. Bistafa, *Annotated Translations of Three of the Euler's Papers on Celestial Mechanics*, Adv. Hist. Stud. **8**, 252 (2019),
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