A Sliding-Block Problem

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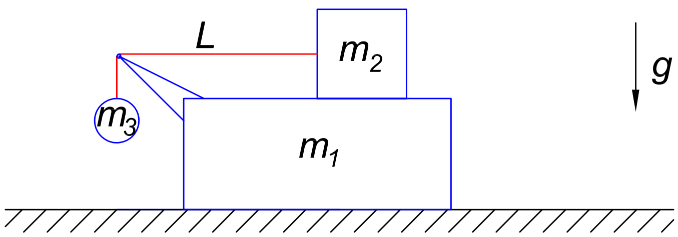
1 Problem

Discuss the motion of a the system sketched below in which a block of mass m_1 slides without friction on a horizonal surface, a block of mass m_2 slides without friction on top of block 1, and mass m_3 is attached to block 2 by a string of length L that passes over a tiny, frictionless pulley supported by block 1.

You may ignore the possibility that block 1 tips, and limit your discussion to motion without oscillation of mass 3, such that the accelerations of the three masses are all constant.

This problem was inspired by [1].

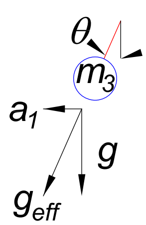


2 Solution

2.1 Uniformly Accelerated Motion

As mass 3 falls, mass 2 accelerates to the left and masses 1 and 3 accelerate to the right, such that the horizontal (x) coordinate of the center of mass of the system remains constant. During this motion, the portion of the string between the pulley and mass 3 takes on a nonzero angle θ to the vertical, and in general this angle is not constant, with mass 3 both translating and rotating.

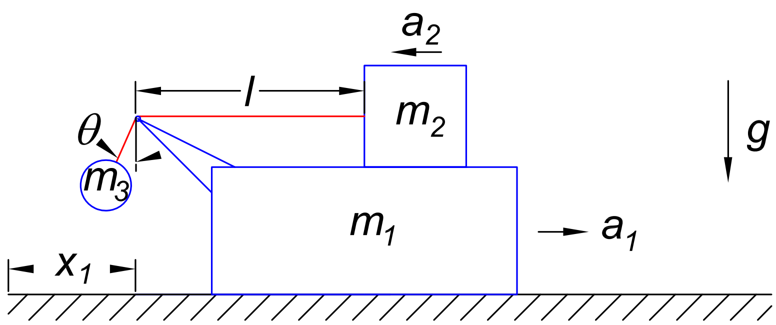
Here, we suppose that the system is launched in such a way that angle θ remains constant (and nonzero), and mass 3 does not rotate as the three masses accelerate.



To determine angle θ, we go to the accelerated frame in which mass 1 is at rest. Then, there exists a "coordinate force" $-m_3\,\mathbf{a}_1$ on mass 3, where $\mathbf{a}_1$ is the acceleration of mass 1 in the inertial lab frame, such that the effective gravity $\mathbf{g}_{\rm eff}$ on mass 3 is as sketched in the figure above. If angle θ is constant and mass 3 does not rotate, then mass 3 moves only along the direction of the string (in the rest frame of mass 1), such that,

$$\tan\theta = \frac{a_1}{g}, \qquad a_1 = g\tan\theta. \tag{1}$$

The rest of the analysis is performed in the lab frame, with notation in the sketch below.



The horizontal coordinate of the (tiny) pulley is x_1, such that,

$$\ddot{x}_1 = a_1 = g\tan\theta \tag{2}$$

is the acceleration of mass 1. We denote the length of the string between the pulley and mass 2 as l, such that the horizontal coordinate of mass 2 is (to within a constant) $x_2 = x_1 + l$, and the acceleration of mass 2 is,

$$\ddot{x}_2 = a_2 = a_1 + \ddot{l} = g\tan\theta + \ddot{l}. \tag{3}$$

The horizontal coordinate of mass 3 is (to within a constant) $x_3 = x_1 - (L - l)\sin\theta$, and,

$$\ddot{x}_3 = a_1 + \ddot{l}\sin\theta = g\tan\theta + \ddot{l}\sin\theta. \tag{4}$$

Since the horizontal position of the center of mass of the system is constant (in the lab frame), the horizontal acceleration of the center of mass is zero,

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_3\ddot{x}_3 = (m_1 + m_2 + m_3)g\tan\theta + (m_2 + m_3\sin\theta)\ddot{l} = 0. \tag{5}$$

To obtain additional relations, we consider the tension $T > 0$ in the string. The acceleration a_2 of block 2 can then be related as,

$$T = -m_2 a_2 = -m_2(g\tan\theta + \ddot{l}). \tag{6}$$

The string acts on the pulley such that the horizontal acceleration of block 1 is related by,

$$m_1 a_1 = m_1 g\tan\theta = T(1 - \sin\theta) = -m_2(g\tan\theta + \ddot{l})(1 - \sin\theta). \tag{7}$$

This determines $\ddot{l}$ to be,

$$\ddot{l} = -g\tan\theta\left(1 + \frac{m_1}{m_2(1 - \sin\theta)}\right). \tag{8}$$

Using this in eq. (5), we obtain an equation for angle θ,

$$m_1 + m_2 + m_3 = (m_2 + m_3 \sin\theta)\left(1 + \frac{m_1}{m_2(1-\sin\theta)}\right), \tag{9}$$

$$\sin^2\theta - \left(2 + \frac{m_1(m_2+m_3)}{m_2 m_3}\right)\sin\theta + 1 = 0. \tag{10}$$

For example, suppose that $m_1 = 2m_2 = 4m_3$. Then, eq. (10) becomes,

$$\sin^2\theta - 8\sin\theta + 1 = 0, \qquad \sin\theta = \frac{8 \pm \sqrt{60}}{2} = 0.127,\ 7.87, \tag{11}$$

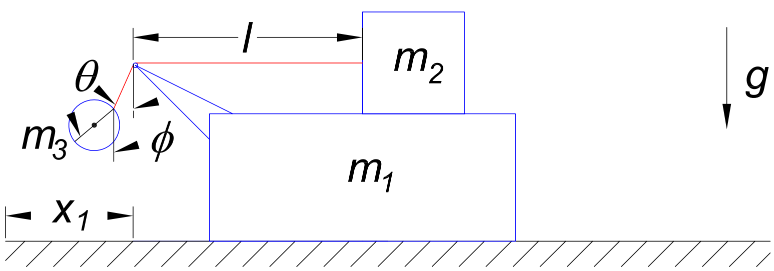
for which the physical solution is $\theta = 7.3^\circ$. The accelerations are,

$$a_1 = 0.128g, \qquad \ddot{l} = -0.421g, \qquad a_2 = -0.293g, \tag{12}$$

$$\ddot{x}_3 = 0.075g, \qquad \ddot{y}_3 = \ddot{l}\cos\theta = -0.417g, \qquad |\mathbf{a_3}| = 0.424g. \tag{13}$$

2.2 Small Oscillations of m_3

We now consider more general motion of the system, in which mass 3 oscillates as well as translates. For this, we suppose the mass 3 is a solid sphere of radius r, and moment of inertia $2m_3 r^2/5$ about its center.



As before, $x_2 = x_1 + l$ to within a constant, so,

$$v_2^2 = \dot{x}_2^2 = \dot{x}_1^2 + \dot{l}^2 + 2\dot{x}_1\dot{l}, \tag{14}$$

while now the coordinates of mass 3 are,

$$x_3 = x_1 - (L-l)\sin\theta - r\sin\phi, \tag{15}$$

$$y_3 = -(L-l)\cos\theta - r\cos\phi, \tag{16}$$

$$\dot{x}_3 = \dot{x}_1 + \dot{l}\sin\theta - (L-l)\,\dot{\theta}\cos\theta - r\,\dot{\phi}\cos\phi, \tag{17}$$

$$\dot{y}_3 = \dot{l}\cos\theta + (L-l)\,\dot{\theta}\sin\theta + r\,\dot{\phi}\sin\phi, \tag{18}$$

$$v_3^2 = \dot{x}_1^2 + \dot{l}^2 + (L-l)^2\dot{\theta}^2 + r^2\dot{\phi}^2 + 2\dot{x}_1\,\dot{l}\sin\theta - 2\dot{x}_1(L-l)\,\dot{\theta}\cos\theta - 2\dot{x}_1\,r\,\dot{\phi}\cos\phi$$
$$+2r\,\dot{l}\,\dot{\phi}\sin(\phi-\theta) - 2r(L-l)\,\dot{\theta}\,\dot{\phi}\cos(\phi-\theta). \tag{19}$$

The potential energy is, to within a constant, $V = -m_3\,g[(L-l)\cos\theta + r\cos\phi]$.

The Lagrangian of the system is,

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}(m_1+m_2+m_3)\,\dot{x}_1^2 + \frac{1}{2}(m_2+m_3)\,\dot{l}^2 + (m_2+m_3\sin\theta)\,\dot{x}_1\,\dot{l} + \frac{m_3}{2}(L-l)^2\,\dot{\theta}^2 \\
&+\frac{m_3}{2}r^2\,\dot{\phi}^2 - m_3\,\dot{x}_1(L-l)\,\dot{\theta}\cos\theta - m_3\,r\,\dot{x}_1\,\dot{\phi}\cos\phi + m_3\,r\,\dot{l}\,\dot{\phi}\sin(\phi-\theta) \\
&+m_3\,r(L-l)\dot{\theta}\,\dot{\phi}\cos(\phi-\theta) + \frac{2m_3\,r^2}{5}\frac{\dot{\phi}^2}{2} + m_3\,g[(L-l)\cos\theta + r\cos\phi]. \qquad (20)
\end{aligned}
$$

The equations of motion for the four coordinates x_1, l, θ and ϕ are,

$$
\begin{aligned}
&(m_1+m_2+m_3)\,\ddot{x}_1 + (m_2+m_3\sin\theta)\,\ddot{l} \\
&+m_3[(L-l)\,\ddot{\theta}\cos\theta - (L-l)\,\dot{\phi}^2\sin\theta - \dot{l}\,\dot{\phi}\cos\theta + r\,\ddot{\phi}\cos\phi - r\,\dot{\phi}^2\sin\phi] = 0, \qquad (21) \\
&(m_2+m_3\sin\theta)\,\ddot{x}_1 + (m_2+m_3)\,\ddot{l} + m_3\,r\,\ddot{\phi}\sin(\phi-\theta) + m_3(L-l)\,\dot{\theta}^2 \\
&+m_3\,r\,\dot{\phi}^2\cos(\phi-\theta) = -m_3\,g\cos\theta, \qquad (22) \\
&-m_3\,\ddot{x}_1(L-l)\cos\theta + m_2(L-l)^2\,\ddot{\theta} + m_3\,r(L-l)\,\ddot{\phi}\cos(\phi-\theta) \\
&-2m_2(L-l)\,\dot{l}\,\dot{\theta} - m_3\,r(L-l)\,\dot{\phi}^2\sin(\phi-\theta) = -m_3\,g(L-l)\sin\theta, \qquad (23) \\
&-m_3\,r\,\ddot{x}_1\cos\phi + m_3\,r\,\ddot{l}\sin(\phi-\theta) + m_3\,r(L-l)\,\ddot{\theta}\cos(\phi-\theta) + 7m_3\,r^2\,\ddot{\phi}/5 \\
&-m_3\,r\,\dot{l}\,\dot{\theta}\cos(\phi-\theta) + m_3\,r(L-l)\,\dot{\theta}^2\sin(\phi-\theta) = -m_3\,g\,r\sin\phi. \qquad (24)
\end{aligned}
$$

For the case analyzed in sec. 2 above, with $\theta = \phi =$ constant, the equations of motion (21)-(24) simplify to,

$$
\begin{aligned}
&(m_1+m_2+m_3)\,\ddot{x}_1 + (m_2+m_3\sin\theta)\,\ddot{l} = 0, \qquad (25) \\
&(m_2+m_3\sin\theta)\,\ddot{x}_1 + (m_2+m_3)\,\ddot{l} = -m_3\,g\cos\theta, \qquad (26) \\
&\ddot{x}_1 = g\tan\theta, \qquad (27) \\
&\ddot{x}_1 = g\tan\phi = g\tan\theta. \qquad (28)
\end{aligned}
$$

Equation (25) is the same as eq. (5), which expresses that the x-coordinate of the center of mass is constant. Equations (27)-(28) are the same as eq. (2), which relates to the effective gravity in the accelerated frame of mass 1. Using these relations in eq. (26), we recover eq. (10) for $\sin\theta$ after some algebra.

For a system that starts from rest at time $t = 0$, we write the solution found in sec. 2 as,

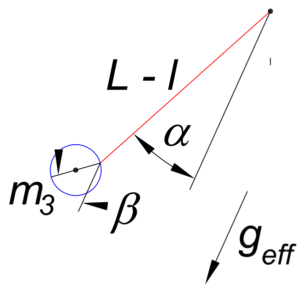
$$
x_{1,0}(t), \qquad l_0(t), \qquad \theta = \phi = \theta_0, \qquad g_{\text{eff},0} = g/\cos\theta_0. \qquad (29)
$$

Returning to the general case, in which mass 3 oscillates and rotates as it falls, we are reminded of Poe's *The Pit and the Pendulum* [2]. The physics of a lengthening pendulum was considered by Lecornu in 1895 [3] and Lord Rayleigh (1902) [4], and that of a (slowly) shortening pendulum was the topic of a famous brief exchange between Lorentz and Einstein at the 1911 Solvay Conference [5], where Einstein's remark anticipated the notion of adiabatic invariance.[1,2]

[1] Einstein's comment, that the ratio E/ν of the energy E of an oscillator to its frequency ν should be constant, may have been motivated by Planck's quantum condition that $E = h\nu$ for an oscillator.

[2] The formal development of the concept of adiabatic invariance is attributed to Ehrenfest [6].

The oscillator motion of mass 3 is approximately that of a compound pendulum subject to a time-dependent effective gravitational acceleration $g_{\rm eff}(t)$, such that the angular frequency ω of oscillation is of order $\sqrt{g_{\rm eff}(t)/[L - l(t)]}$. Here, we make only a kind of adiabatic approximation that the motion of mass 3 at time t is as if $g_{\rm eff}$ and $L-l$ have constant values, namely those at time t. This case is represented in the figure below.



The Lagrangian for this subsystem, with angular coordinates α and β, is,

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{m_3}{2}(L-l)^2\,\dot\alpha^2 + \frac{m_3}{2}\frac{7r^2}{5}\,\dot\beta^2 + m_3\,r(L-l)\,\dot\alpha\,\dot\beta\cos(\alpha-\beta) \\
& + m_3\,g_{\rm eff}[(L-l)\cos\alpha + r\cos\beta],
\end{aligned}
\tag{30}
$$

for which the equation of motion are,

$$
(L-l)^2\,\ddot\alpha + r(L-l)\,\ddot\beta\cos(\alpha-\beta) - r(L-l)\,\dot\beta^2\sin(\alpha-\beta) \;=\; -g_{\rm eff}(L-l)\sin\alpha, \tag{31}
$$

$$
\frac{7r^2}{5}\,\ddot\beta + r(L-l)\,\ddot\alpha\cos(\alpha-\beta) + r(L-l)\,\dot\alpha^2\sin(\alpha-\beta) \;=\; -g_{\rm eff}\,r\sin\beta. \tag{32}
$$

We next consider small oscillations, $\alpha = \alpha_0\,e^{i\omega t}$, $\beta = \beta_0\,e^{i\omega t}$, for which the equations of motion simply further to,

$$
[(L-l)\,\omega^2 - g_{\rm eff}]\,\alpha_0 + r\,\omega^2\,\beta_0 \;=\; 0, \tag{33}
$$

$$
(L-l)\,\omega^2\,\alpha_0 + \left(\frac{7r}{5}\,\omega^2 - g_{\rm eff}\right)\beta_0 \;=\; 0. \tag{34}
$$

For a solution to exist, ω must satisfy,

$$
\frac{2r(L-l)}{5}\,\omega^4 - g_{\rm eff}\left(L - l + \frac{7r}{5}\right)\omega^2 + g_{\rm eff}^2 = 0, \tag{35}
$$

which implies that there are two modes of oscillation, with angular frequencies,

$$
\omega = \sqrt{\frac{5g_{\rm eff}}{4\,r(L-l)}\left(L - l + \frac{7r}{5} \pm \sqrt{(L-l)^2 + \frac{6\,r(L-l)}{5} + \frac{49r^2}{25}}\right)}. \tag{36}
$$

As usual for a compound pendulum, the small angles α and β have the same sign for the lower frequency of oscillation, and opposite signs for the higher frequency.

Our adiabatic approximation is that the time-dependent frequencies $\omega(t)$ are obtained by using $l(t) = l_0(t)$, and $g_{\rm eff}(t) = g_{\rm eff,0}$ from eq. (29) in eq. (36).

References

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