Cylinder Rolling inside a Fixed Cylinder

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(August 13, 2013)

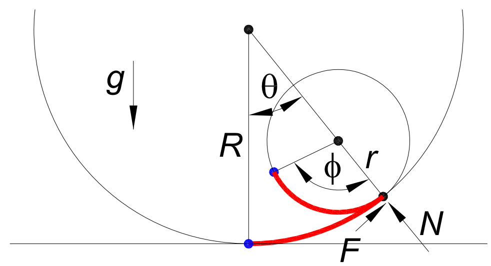
1 Problem

Discuss the motion of a cylinder that rolls without slipping inside a fixed cylinder whose axis is horizontal.

2 Solution

This problem is a simpler version of the case of one cylinder rolling inside another rolling cylinder [1].

A cylinder (or sphere) of mass m, radius r, and moment of inertia kmr^2 about its center[1] is in contact with a cylindrical surface of radius $R > r$ as sketched below.



The center of the cylinder of radius r makes angle θ with respect to a vertical through the center(line) of the cylinder of radius R.

The condition of rolling without slipping for the inner cylinder is that when its center has rolled from $\theta = 0$ to θ, the initial point of contact of the two cylinders makes angle ϕ with respect to the present point of contact, which is at distance $r\theta$ from the initial point of contact along the surface of the outer cylinder. That is,

$$r\phi = R\theta. \tag{1}$$

The angular velocity ω of the inner, rolling cylinder is not $\dot{\phi}$, but is to be measured with respect to a fixed direction in the lab frame, such as the vertical. That is,

$$\omega = \dot{\phi} - \dot{\theta} = \dot{\theta}\frac{R-r}{r}\,. \tag{2}$$

The angular acceleration α of the inner cylinder is then,

$$\alpha = \dot{\omega} = \ddot{\theta}\frac{R-r}{r}\,. \tag{3}$$

[1] $k = 1$ for a cylindrical shell, $k = 1/2$ for a uniform cylinder, and $k = 2/5$ for a uniform sphere, *etc.*

The potential energy of the inner cylinder, taken to be zero when $\theta = 0$, is,

$$\text{PE} = mg(R - r)(1 - \cos\theta). \tag{4}$$

The kinetic energy of the inner cylinder is, whose center-of-mass velocity is $v = (R - r)\dot{\theta}$

$$\text{KE} = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mr^2\dot{\theta}^2}{2} + \frac{kmr^2\dot{\theta}^2}{2}\left(\frac{R-r}{r}\right)^2 = \frac{m\dot{\theta}^2}{2}[r^2 + k(R - r)^2]. \tag{5}$$

Assuming no slipping of the inner cylinder, the conserved energy E of the system, with initial velocity v_0 when $\theta = 0$, is,

$$E = \text{KE} + \text{PE} = \frac{mv_0^2}{2} = \frac{m\dot{\theta}^2}{2}(1 + k)(R - r)^2 + mg(R - r)(1 - \cos\theta)$$
$$= \frac{mv^2}{2}(1 + k) + mg(R - r)(1 - \cos\theta). \tag{6}$$
$$v^2 = \frac{m}{1 + k}[v_0^2 - 2g(R - r)(1 - \cos\theta)]. \tag{7}$$

The equation of motion in angle θ follows from the time derivative of eq. (6),

$$\ddot{\theta} = -\frac{g\sin\theta}{(1 + k)(R - r)}. \tag{8}$$

Note that $\ddot{\theta}$ is negative for positive θ, and hence the angular acceleration α of eq. (3) is also negative for positive θ; *i.e.*, counterclockwise in the figure on p. 1. Hence, the force F of friction at the contact point is directed up the slope of the outer cylinder.

2.1 Small Oscillation about $\theta = 0$

For small θ, the equation of motion (8) has the form,

$$\ddot{\theta} \approx -\frac{g}{(1 + k)(R - r)}\theta, \tag{9}$$

which has the form of simple-harmonic oscillation, $\theta = \theta_0 \sin\Omega t$ ($\Rightarrow v_0 = (R - r)\,\Omega\,\theta_0$), with angular frequency Ω given by,

$$\Omega = \sqrt{\frac{g}{(1 + k)(R - r)}}. \tag{10}$$

2.2 At What Angle Does the Inner Cylinder Slip?

If v_0 is large enough that angle θ reaches 90°, then the inner cylinder slips/falls at that angle. The condition for this follows from eq. (6) as,

$$v_0^2(\theta \geq 90^\circ) \geq 2g(R - r). \tag{11}$$

When $\theta_{\rm max} < 90°$, the inner cylinder will slip if the friction force F between the inner and outer cylinders exceeds μN, where μ is the coefficient of static friction between the inner and outer cylinder.

The normal (radial) force is related to the centripetal acceleration $mv^2/(R-r)$ of the inner cylinder by, recalling eq. (7),

$$N - mg\sin\theta = \frac{mv^2}{R-r}, \qquad N = mg\sin\theta + \frac{m}{1+k}\frac{v_0^2 - 2g(R-r)(1-\cos\theta)}{R-r}. \tag{12}$$

The friction force F must be large enough to satisfy the torque equation about the center of mass of the inner cylinder,

$$\tau_{\rm cm} = rF = -I\alpha = -kmr^2\ddot\theta\frac{R-r}{r} = kmr(R-r)\frac{g\sin\theta}{(1+k)(R-r)}\frac{R-r}{r} = \frac{kmg(R-r)\sin\theta}{1+k}, \tag{13}$$

$$F = \frac{kmg(R-r)\sin\theta}{(1+k)r}, \tag{14}$$

recalling eqs. (3) and (9). The slip angle $\theta_{\rm slip}$, if it exists, is given by the condition, $F = \mu N$,

$$\frac{k(R-r)g\sin\theta_{\rm slip}}{(1+k)r} = \mu\left[g\sin\theta_{\rm slip} + \frac{v_0^2 - 2g(R-r)(1-\cos\theta_{\rm slip})}{(1+k)(R-r)}\right]. \tag{15}$$

For the inner cylinder to reach angle $\theta \le 90°$ without slipping, the coefficient of friction must satisfy,

$$\mu(\theta) > \frac{k(R-r)g\sin\theta_{\rm slip}}{(1+k)r} \Big/ \left[g\sin\theta_{\rm slip} + \frac{v_0^2 - 2g(R-r)(1-\cos\theta_{\rm slip})}{(1+k)(R-r)}\right] = \frac{k}{1+k}\frac{R-r}{r}. \tag{16}$$

using $\theta_{\rm slip} = 90°$ and $v_0^2 = 2g(R-r)$ as the minimum to reach $\theta = 90°$.

For larger ratio R/r, the coefficient of static friction μ must be larger for the inner cylinder to reach $\theta = 90°$, *i.e.*, the height of the center of the outer cylinder.

References

[1] K.T. McDonald, *Cylinder Rolling inside Another Rolling Cylinder*, (Oct. 23, 2014), http://kirkmcd.princeton.edu/examples/2cylinders_in.pdf