# How Circularly Polarized is the Forward Radiation from a Helical Undulator?

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(October 10, 2008)

The radiation emitted in the forward direction ($\theta = 0$) by an electron that moves along the axis of an ideal helical undulator is purely circularly polarized.[1] Here, we consider to what extent this ideal might not be realized in practice.

If the trajectory of an electron inside the undulator is a circular helix, then the radiation emitted in the forward direction will be purely circularly polarized (with handedness opposite to that of the helix, according to an observer who looks into the wave). For the motion to be purely helical, the motion transverse to the (average) trajectory of the electron must be purely circular, which requires that the magnetic field in direction transverse to the trajectory must vary sinusoidally with position along the trajectory.

If it useful to note that the so-called formation length for undulator radiation [1, 2] is essentially a single period $\lambda_u$ of the undulator, so only departures from ideal behavior over a single period of the undulator affects the polarization of the radiation.

If the windings of the undulator are ideal helices, then the magnitude $B$ of the transverse magnetic field does increase slightly with distance $r \lesssim ƛ_u = \lambda_u/2\pi$ from the axis [2],

$$B(r) = B_0 \left[ 1 + \frac{1}{8} \left( \frac{r}{ƛ_u} \right)^2 + \cdots \right], \qquad (1)$$

but for a any trajectory parallel to the undulator axis (and inside the undulator), the longitudinal variation of the field remains sinusoidal. Also note that the radius $R$ of the helix of an electron with $\gamma = 1/\sqrt{1 - v^2/c^2}$ in a helical magnetic field $B$ is [1]

$$R = \frac{K\lambda_u}{2\pi\gamma}, \qquad (2)$$

where

$$K = \frac{eB\lambda_u}{2\pi mc^2} \qquad (3)$$

is the dimensionless undulator-strength parameter (typically of order 1). Thus, for ultra-relativistic electrons the variation of the magnetic field is negligible over the trajectory of an electron at distance $r \lesssim ƛ$ from the undulator axis. Hence, a transverse offset $r \lesssim ƛ$ of the electron trajectory through the undulator makes no change in the polarization of the radiation.

An electron that moves of an average trajectory that makes a small angle $\alpha$ to the axis of the undulator experiences a local magnetic field that rotates with the undulator period

[1] If the windings of a helical undulator are right-handed, then the electric field vector of the radiation emitted in the forward direction by an electron the moves along the axis of the undulator has the form of a right-handed screw at any given time. In this case the angular momentum vector of the radiation is in the direction of motion of the wave and we say that the longitudinal polarization of wave photons is $P_\gamma = +1$.

$\lambda_u$ but whose magnitude varies slowly according to eq. (1). To a good approximation the trajectory of the electron is helical, but with a radius given by eqs. (2)-(3) using the local value of the magnetic field $B(r)$. The acceleration vector is still proportional $Re(\hat{\mathbf{x}} \pm i\,\hat{\mathbf{y}})$, so the radiation emitted in the (tilted) direction of the trajectory is still purely circularly polarized.[2]

While the polarization of the forward radiation from a tilted electron trajectory is still circularly polarized, the energy of the photons emitted in this direction is affected by the tilt of the trajectory; the $K$ factor increases off axis, which decreases the energy of the forward photons by the factor

$$\frac{1+K_0^2}{1+K^2(r)} \approx \frac{1+K_0^2}{1+K_0^2+\alpha^2 K_0^2 z^2/4\bar{\lambda}_u^2} \approx 1 - \frac{K_0^2}{1+K_0^2}\frac{\alpha^2 z^2}{4\bar{\lambda}_u^2}, \tag{8}$$

recalling eqs. (1) and (3) and that energy $E_0$ of the forward photons scales as $1/(1+K^2)$ [1]. For an undulator of length $L$ the average decrease in energy of the forward photons is

$$\frac{\Delta E_0}{E_0} \approx -\frac{1}{L}\int_{L/2}^{L/2} \frac{K_0^2}{1+K_0^2}\frac{\alpha^2 z^2}{4\bar{\lambda}_u^2}\,dz = -\frac{K_0^2}{1+K_0^2}\frac{\alpha^2 L^2}{48\bar{\lambda}_u^2}\,. \tag{9}$$

In practice, the maximum tilt angle such that the electron beam would not scrape the undulator is $\alpha_{\max} \approx 2L/\bar{\lambda}_u$, in which case

$$\frac{\Delta E_{0,\max}}{E_0} \approx -\frac{1}{12}\frac{K_0^2}{1+K_0^2}\,. \tag{10}$$

As an example, in SLAC experiment E166 [3] the left-handed helical undulator was 1 m long, with $\lambda_u = 2.54$ mm, and $K_0 \approx 0.17$, for which $\Delta E_{0,\max}/E_0 \approx 0.002$.

Similarly, even if the helical winding of the undulator does not have a uniform pitch, the forward radiation is still circularly polarized because the electron trajectory remains helical, although its spectrum is slightly altered. The energy of the forward photons scales inversely with the period of the undulator, so if, for example, a particular period of the undulator was longer than nominal by a factor 0.001, then the energy of forward photons emitted by this period of the undulator would be lower than nominal by the same factor 0.001.

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[2] Consider an average trajectory $x = \alpha z \approx \alpha ct$, $y = 0$, so the helical trajectory in a right-handed undulator is

$$x = \alpha z + R(z)\cos(z/\bar{\lambda}_u) = \alpha ct + R(t)\cos(\omega_u t), \qquad y = R(z)\sin(z/\bar{\lambda}_u) = R(t)\sin(\omega_u t), \tag{4}$$

where

$$R(z) = R_0(1+\alpha^2 z^2/8\bar{\lambda}_u^2) = R(t) = R_0(1+\alpha^2\omega_u^2 t^2/8), \tag{5}$$

and $\omega_u = c/\bar{\lambda}_u$. The acceleration of the electron is

$$\ddot{x} = Re[(\ddot{R} - \omega_u^2 R - 2i\omega_u \dot{R})e^{-i\omega_u t}], \qquad \ddot{y} = Re[i(\ddot{R} - \omega_u^2 R - 2i\omega_u \dot{R})e^{-i\omega_u t}], \tag{6}$$

so the acceleration vector can be written

$$\ddot{\mathbf{x}} = (\ddot{R} - \omega_u^2 R - 2i\omega_u \dot{R})(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})e^{-i\omega_u t}. \tag{7}$$

The most plausible effect that could result in a departure of the forward radiation from pure circular polarization is that the undulator is wound on an elliptical rather than a circular tube. This would result in the trajectory of an electron being an elliptical helix, say with semimajor axis $R_x = R_0(1+\epsilon)$ and semiminor axis $R_y = R_0(1-\epsilon)$. For small $\epsilon$, the $x$ and $y$ coordinates of the electron trajectory that lies (on average) along the axis of a right-handed undulator axis can be written

$$x = R_0(1+\epsilon)\cos(\omega_u t), \qquad y = R_0(1-\epsilon)\sin(\omega_u t), \tag{11}$$

so the acceleration vector can be written as the real part of

$$\ddot{\mathbf{x}} = R_0[\hat{\mathbf{x}} + i\,\hat{\mathbf{y}} + \epsilon(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}})]e^{-i\omega_u t}. \tag{12}$$

Hence, the forward radiation has an admixture of amplitude $\epsilon$ of circular polarization opposite to the nominal. The longitudinal polarization of the forward radiation would be lower than nominal by the factor $(1-\epsilon)/(1+\epsilon)$.

The radius of curvature of the helical trajectory scales inversely with the magnetic field, while the magnetic field scales inversely with the radius of the windings [2], so the parameter $\epsilon$ also describes the ellipticity of the undulator windings. Hence, any departure from unity of the polarization of the forward photons provides a direct measure of the ellipticity of the undulator windings.

# References


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