Effective Center, Obliquity Factor and Solid Angle of R5912 Photomultiplier Tubes

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(Version 4, July 23, 2011)

Abstract

The average location of a photoelectron detected by an R5912 photomultiplier tube (PMT) is at a distance ≈ 2 cm in from the intercept of the axis and the front face of the PMT, and the area of the photocathode as projected onto a plane perpendicular to the direction of an incident photon can be described by an obliquity factor,

$$\alpha(\theta) = \begin{cases} \cos\theta & (\cos\theta > \cos\theta_0), \\ 0.19 + 1.71\cos^2\theta - 0.95\cos^2\theta & (\cos\theta < \cos\theta_0), \end{cases} \tag{1}$$

where θ is the polar angle between the axis of the PMT and the direction of the photon and $\cos\theta_0 \approx 0.69$. The solid angle subtended by the photocathode with respect to a point at distance r from the average location of the PMT is,

$$\Omega \approx 2\pi\left(1 - \frac{r}{\sqrt{r^2 + \alpha r_1^2}}\right), \tag{2}$$

where $r_1 = 9.5$ cm.

1 Introduction

As reported by Hamamatsu, the photocathode of their R5912 photomultiplier tubes (PMTs) has a radius of curvature of $r_0 = 13.1$ cm and constitutes of spherical cap of chord $2\,r_1 \gtrsim$ 19 cm, as shown on the next page. The polar angle of the edge of the photocathode is $\theta_0 \gtrsim \sin^{-1}(r_1/r_0) = 46.5^\circ = 0.81$ rad. The distance from the center of curvature to the plane of the cap is $r_2 = r_0\cos\theta_0 \lesssim 9$ cm.

2 Average x, y and z of Photoelectrons

It will often be convenient to represent the coordinates of the photoelectrons j generated at the photocathode of PMT of index i by a single point $(\langle x_j\rangle, \langle y_j\rangle, \langle z_j\rangle) = (x_i, y_i, z_i)$.

It is appealing to take this point as lying along the symmetry axis of the PMT, although this is strictly correct only for photons with polar angle $\theta = 0$ relative to that axis.

For photons with $\theta = 0$ in a cylindrical coordinate system ρ, ϕ, z' with z'-axis along that of the PMT and origin at the center of curvature of the photocathode, the average z' is,

$$\langle z'\rangle = \frac{\int_0^{r_1} \rho\,d\rho\,z'(\rho)}{\int_0^{r_1}\rho\,d\rho} = \frac{2}{r_1^2}\int_0^{r_1}\rho\,d\rho\,\sqrt{r_0^2 - \rho^2} = \frac{2}{3r_0^2\sin^2\theta_0}[r_0^3 - (r_0^2 - r_1^2)^{3/2}]$$

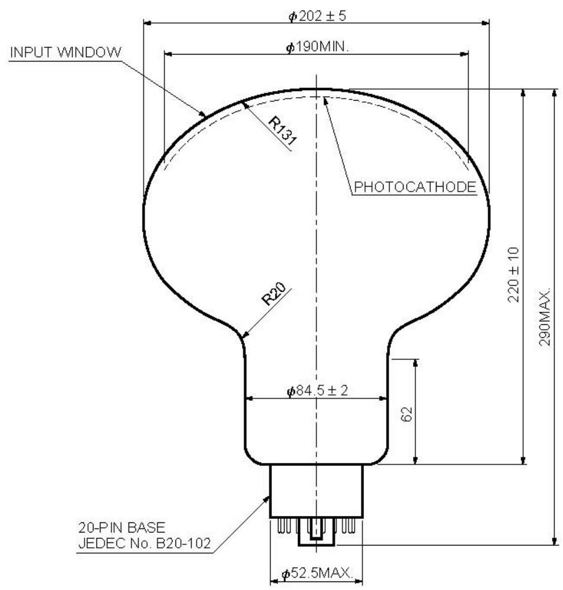
$$= \frac{2r_0}{3\sin^2\theta_0}(1-\cos^3\theta_0) = 11.2 \text{ cm}, \tag{3}$$

noting that the surface of the photocathode is given by $z'(\rho) = \sqrt{r_0^2 - \rho^2}$. At the other extreme, $\theta = 90°$, the average z' is,

$$\langle z' \rangle = \frac{\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz'\, z'}{\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz'} = \frac{\int_0^{r_1} d\rho (r_0^2 - \rho^2 - r_2^2)/2}{\int_0^{r_1} d\rho \left(\sqrt{r_0^2-\rho^2} - r_2\right)} = \frac{r_0^2 r_1/2 - r_1^3/6 - r_1 r_2^2/2}{(r_0^2\theta_0 - r_1 r_2)/2}$$

$$= r_0 \frac{2\sin^3\theta_0}{3(\theta_0 - \cos\theta_0\sin\theta_0)} = 10.1 \text{ cm}. \tag{4}$$

I propose taking $\langle z \rangle$ for the photoelectrons to be on the axis of the PMT and 2 cm in from the front face (rather than distance $r_0 = 13.1$ cm in, as perhaps presently used in the analysis).



3 Obliquity Factor

Photons with polar angle θ less than $90° - \theta_0 = 43.5°$ can strike the photocathode anywhere, and the projection of the area of the photocathode onto a plane perpendicular to the photon's trajectory is,

$$A_{\text{proj}}(\theta) = A\cos\theta, \tag{5}$$

where $A = \pi r_1^2 = 284$ cm^2 is the area of the plane of the cap. For θ greater than $90° - \theta_0 = 43.5°$ the projected area has a complicated behavior, reaching a minimum for $\theta = 90°$,

$$A_{\text{proj}}(\theta = 90°) = 2\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz' = r_0^2\theta_0 - r_1 r_2 = A\frac{\theta_0 - \cos\theta_0\sin\theta_0}{\pi\sin^2\theta_0} = 0.19A \equiv aA. \tag{6}$$

For θ near $90°$ the projected area varies slowly with angle, so I suppose that it can be represented for $0 < \cos\theta < \cos\theta_0$ with the form,

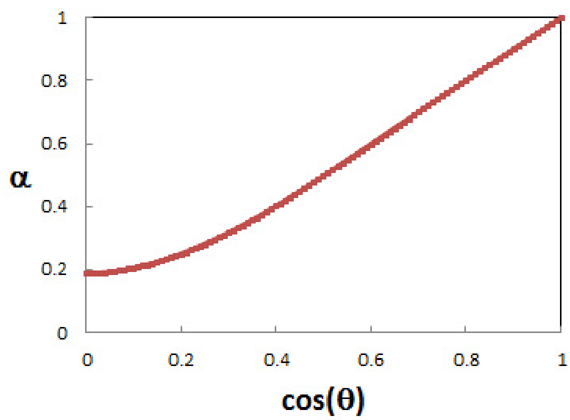
$$A_{\text{proj}}(0 < \cos\theta < \cos\theta_0) = A(a + b\cos^2\theta + c\cos^3\theta), \tag{7}$$

where b and chosen such that $A_{\rm proj}(\cos\theta_0) = A\cos\theta_0$ and $dA_{\rm proj}(\cos\theta_0)/d\cos\theta = A$. Hence,

$$b = \frac{2\cos\theta_0 - 3a}{\cos^2\theta_0} = 1.71, \qquad c = \frac{2a - \cos\theta_0}{\cos^3\theta_0} = -0.95. \tag{8}$$

Introducing the obliquity factor $\alpha = A_{\rm proj}/A$ we have,

$$\alpha(\theta) = \begin{cases} \cos\theta & (\cos\theta > \cos\theta_0), \\ a + b\cos^2\theta + c\cos^2\theta & (\cos\theta < \cos\theta_0). \end{cases} \tag{9}$$



For completeness, we note that if we represent the average location of PMT i by (x_i, y_i, z_i) and a unit vector along its axis by $(e_{ix}, e_{iy}, 0)$, then a photon that originates at point (x, y, z) has polar angle with respect to this PMT given by,

$$\cos\theta_i = \frac{e_{ix}(x - x_i) + e_{iy}(y - y_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}. \tag{10}$$

4 Solid Angle

The number of photoelectrons generated in a PMT by a point source at distance r from the effective center of the PMT is proportional to the solid angle subtended by the PMT. The PMT has projected area $\pi\alpha r_1^2$ which is equivalent to a circle of radius $r_{\rm eff} = r_1\sqrt{\alpha}$. The equivalent circle subtends a cone of half angle $\theta_{\rm eff}$ related by $\tan\theta_{\rm eff} = r_{\rm eff/r}$ The solid angle Ω subtended by the equivalent circle is,

$$\Omega = 2\pi(1 - \cos\theta_{\rm eff}) = 2\pi\left(1 - \frac{r}{\sqrt{r^2 + r_{\rm eff}^2}}\right) = 2\pi\left(1 - \frac{r}{\sqrt{r^2 + \alpha r_1^2}}\right). \tag{11}$$

Only when $r_1 \ll r$ does the solid angle have the simple form $\pi\alpha r_1^2/r^2$.

Approximating the projected area of the photocathode by an equivalent circle would lead to some error in the solid angle Ω for $\theta \approx 90°$ and $r \approx r_1$. However, such a source region is in the outer LAB buffer, where there is no scintillation. For scintillation sources close to a photomultiplier, $\theta \lesssim \theta_0$, the use of an equivalent circle is a reasonable approximation.