Effective Center, Obliquity Factor and Solid Angle of R5912 Photomultiplier Tubes

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(Version 4, July 23, 2011)

Abstract

The average location of a photoelectron detected by an R5912 photomultiplier tube (PMT) is at a distance ≈ 2 cm in from the intercept of the axis and the front face of the PMT, and the area of the photocathode as projected onto a plane perpendicular to the direction of an incident photon can be described by an obliquity factor,

$$\alpha(\theta) = \begin{cases} \cos\theta & (\cos\theta > \cos\theta_0), \\ 0.19 + 1.71\cos^2\theta - 0.95\cos^2\theta & (\cos\theta < \cos\theta_0), \end{cases} \tag{1}$$

where θ is the polar angle between the axis of the PMT and the direction of the photon and $\cos\theta_0 \approx 0.69$. The solid angle subtended by the photocathode with respect to a point at distance r from the average location of the PMT is,

$$\Omega \approx 2\pi\left(1 - \frac{r}{\sqrt{r^2 + \alpha r_1^2}}\right), \tag{2}$$

where $r_1 = 9.5$ cm.

1 Introduction

As reported by Hamamatsu, the photocathode of their R5912 photomultiplier tubes (PMTs) has a radius of curvature of $r_0 = 13.1$ cm and constitutes of spherical cap of chord $2\,r_1 \gtrsim 19$ cm, as shown on the next page. The polar angle of the edge of the photocathode is $\theta_0 \gtrsim \sin^{-1}(r_1/r_0) = 46.5^\circ = 0.81$ rad. The distance from the center of curvature to the plane of the cap is $r_2 = r_0\cos\theta_0 \lesssim 9$ cm.

2 Average x, y and z of Photoelectrons

It will often be convenient to represent the coordinates of the photoelectrons j generated at the photocathode of PMT of index i by a single point $(\langle x_j\rangle, \langle y_j\rangle, \langle z_j\rangle) = (x_i, y_i, z_i)$.

It is appealing to take this point as lying along the symmetry axis of the PMT, although this is strictly correct only for photons with polar angle $\theta = 0$ relative to that axis.

For photons with $\theta = 0$ in a cylindrical coordinate system ρ, ϕ, z' with z'-axis along that of the PMT and origin at the center of curvature of the photocathode, the average z' is,

$$\langle z'\rangle = \frac{\int_0^{r_1} \rho\, d\rho\, z'(\rho)}{\int_0^{r_1} \rho\, d\rho} = \frac{2}{r_1^2}\int_0^{r_1} \rho\, d\rho\,\sqrt{r_0^2 - \rho^2} = \frac{2}{3r_0^2\sin^2\theta_0}[r_0^3 - (r_0^2 - r_1^2)^{3/2}]$$

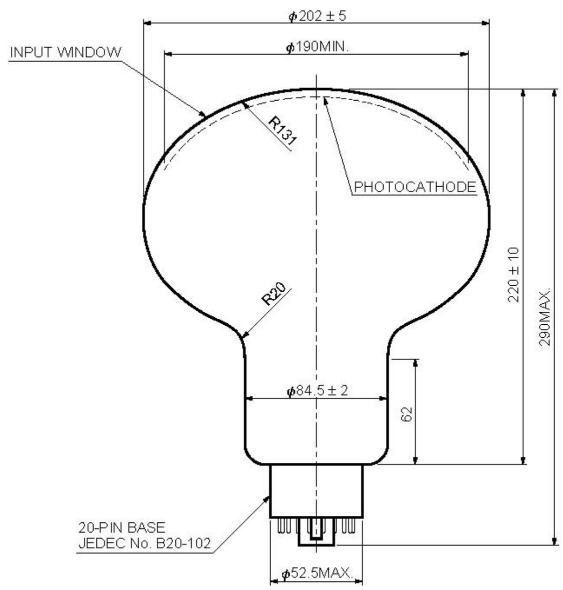
$$= \frac{2r_0}{3\sin^2\theta_0}(1-\cos^3\theta_0) = 11.2 \text{ cm}, \tag{3}$$

noting that the surface of the photocathode is given by $z'(\rho) = \sqrt{r_0^2 - \rho^2}$. At the other extreme, $\theta = 90°$, the average z' is,

$$\langle z' \rangle = \frac{\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz'\, z'}{\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz'} = \frac{\int_0^{r_1} d\rho (r_0^2 - \rho^2 - r_2^2)/2}{\int_0^{r_1} d\rho \left(\sqrt{r_0^2-\rho^2} - r_2\right)} = \frac{r_0^2 r_1/2 - r_1^3/6 - r_1 r_2^2/2}{(r_0^2\theta_0 - r_1 r_2)/2}$$

$$= r_0 \frac{2\sin^3\theta_0}{3(\theta_0 - \cos\theta_0 \sin\theta_0)} = 10.1 \text{ cm}. \tag{4}$$

I propose taking $\langle z \rangle$ for the photoelectrons to be on the axis of the PMT and 2 cm in from the front face (rather than distance $r_0 = 13.1$ cm in, as perhaps presently used in the analysis).



3 Obliquity Factor

Photons with polar angle θ less than $90° - \theta_0 = 43.5°$ can strike the photocathode anywhere, and the projection of the area of the photocathode onto a plane perpendicular to the photon's trajectory is,

$$A_{\text{proj}}(\theta) = A\cos\theta, \tag{5}$$

where $A = \pi\, r_1^2 = 284$ cm^2 is the area of the plane of the cap. For θ greater than $90° - \theta_0 = 43.5°$ the projected area has a complicated behavior, reaching a minimum for $\theta = 90°$,

$$A_{\text{proj}}(\theta = 90°) = 2\int_0^{r_1} d\rho \int_{r_2}^{z'(\rho)} dz' = r_0^2\theta_0 - r_1 r_2 = A\frac{\theta_0 - \cos\theta_0\sin\theta_0}{\pi\sin^2\theta_0} = 0.19A \equiv aA. \tag{6}$$

For θ near $90°$ the projected area varies slowly with angle, so I suppose that it can be represented for $0 < \cos\theta < \cos\theta_0$ with the form,

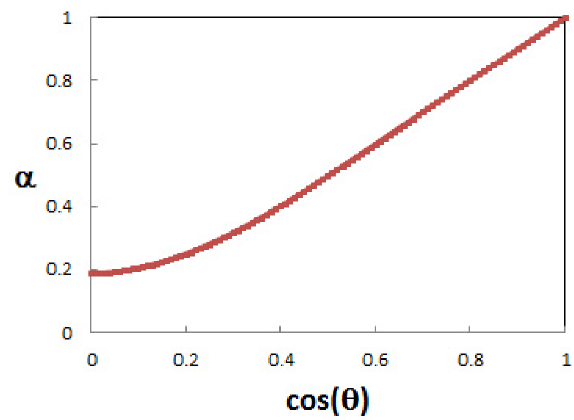
$$A_{\text{proj}}(0 < \cos\theta < \cos\theta_0) = A(a + b\cos^2\theta + c\cos^3\theta), \tag{7}$$

where b and chosen such that $A_{\rm proj}(\cos\theta_0) = A\cos\theta_0$ and $dA_{\rm proj}(\cos\theta_0)/d\cos\theta = A$. Hence,

$$b = \frac{2\cos\theta_0 - 3a}{\cos^2\theta_0} = 1.71, \qquad c = \frac{2a - \cos\theta_0}{\cos^3\theta_0} = -0.95. \tag{8}$$

Introducing the obliquity factor $\alpha = A_{\rm proj}/A$ we have,

$$\alpha(\theta) = \begin{cases} \cos\theta & (\cos\theta > \cos\theta_0), \\ a + b\cos^2\theta + c\cos^2\theta & (\cos\theta < \cos\theta_0). \end{cases} \tag{9}$$



For completeness, we note that if we represent the average location of PMT i by (x_i, y_i, z_i) and a unit vector along its axis by $(e_{ix}, e_{iy}, 0)$, then a photon that originates at point (x, y, z) has polar angle with respect to this PMT given by,

$$\cos\theta_i = \frac{e_{ix}(x - x_i) + e_{iy}(y - y_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}. \tag{10}$$

4 Solid Angle

The number of photoelectrons generated in a PMT by a point source at distance r from the effective center of the PMT is proportional to the solid angle subtended by the PMT. The PMT has projected area $\pi\alpha r_1^2$ which is equivalent to a circle of radius $r_{\rm eff} = r_1\sqrt{\alpha}$. The equivalent circle subtends a cone of half angle $\theta_{\rm eff}$ related by $\tan\theta_{\rm eff} = r_{\rm eff/r}$ The solid angle Ω subtended by the equivalent circle is,

$$\Omega = 2\pi(1 - \cos\theta_{\rm eff}) = 2\pi\left(1 - \frac{r}{\sqrt{r^2 + r_{\rm eff}^2}}\right) = 2\pi\left(1 - \frac{r}{\sqrt{r^2 + \alpha r_1^2}}\right). \tag{11}$$

Only when $r_1 \ll r$ does the solid angle have the simple form $\pi\alpha r_1^2/r^2$.

Approximating the projected area of the photocathode by an equivalent circle would lead to some error in the solid angle Ω for $\theta \approx 90°$ and $r \approx r_1$. However, such a source region is in the outer LAB buffer, where there is no scintillation. For scintillation sources close to a photomultiplier, $\theta \lesssim \theta_0$, the use of an equivalent circle is a reasonable approximation.