Reconstruction of Energy and Position in the Antineutrino Detectors

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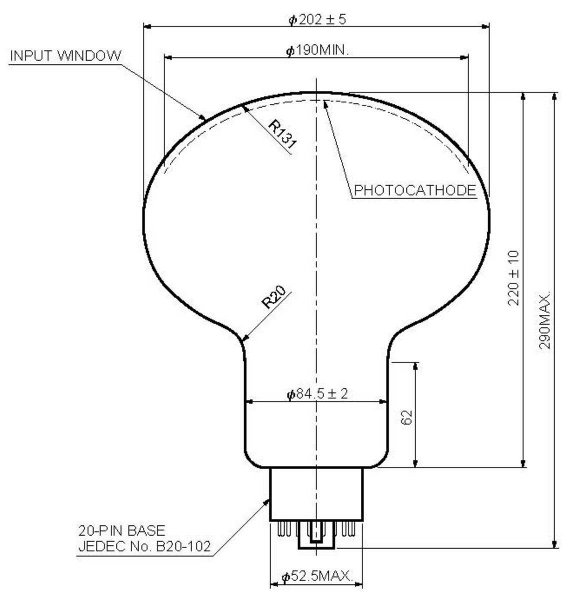
Abstract

A χ^2 algorithm is proposed for simultaneous reconstruction of energy and position of a localized event in the antineutrino detectors.

We consider a localized deposition of energy E at a position $\mathbf{x} = (x, y, z)$ inside one of the antineutrino detectors. Each of the 192 photomultiplier tubes (PMTs) detects n_i photoelectrons at a representative position (x_i, y_i, z_i), along the axis of the PMT, some 2 cm from its front face [1].



We suppose that the number of detected photoelectrons is related to the energy and position of the source by,

$$n_i = \frac{k_i N A E}{4\pi R_i^2}, \qquad \text{where} \qquad R_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2, \tag{1}$$

where k_i is a (possibly time-dependent) coefficient that reflects quantum efficiency of PMT i, N is the number of scintillation photons per unit of deposited energy, $A = \pi r_1^2$ is the area of the PMT photocathode, $r_1 = 9.5$ cm, and the geometric acceptance of photons by a PMT is assumed to fall off with the square of the distance to the source. *In the first approximation, all k_i are the same, about 0.2.*

We form the χ^2 for the observed data $\{n_i\}$ and the parameters x, y, z and E that are to be determined,

$$\chi^2 = \sum_{i=1}^{192} \frac{\left(n_i - \frac{k_i N A E}{4\pi R_i^2}\right)^2}{n_i}, \tag{2}$$

where the standard error for the observation of n_i photoelectrons is taken to be $\sqrt{n_i}$, and terms for which $n_i = 0$ are omitted from the summation.

The best-fit parameters minimize the χ^2, which could be performed by a program such as MINUIT. For a less compute-intensive approach we might consider an iterative procedure, first noting that minimizing the χ^2 implies that,

$$0 = -\frac{2\pi}{NA}\frac{\partial\chi^2}{\partial E} = \sum_i \frac{\frac{k_i}{R_i^2}\left(n_i - \frac{k_i NAE}{4\pi R_i^2}\right)}{n_i}, \tag{3}$$

$$E = \frac{4\pi}{NA}\frac{\sum_i \frac{k_i}{R_i^2}}{\sum_i \frac{k_i^2}{n_i R_i^4}}, \tag{4}$$

$$0 = \frac{\pi}{NAE}\frac{\partial\chi^2}{\partial x} = \sum_i \frac{\frac{k_i(x-x_i)}{R_i^4}\left(n_i - \frac{k_i NAE}{4\pi R_i^2}\right)}{n_i}, \tag{5}$$

$$x = \frac{4\pi}{NA}\frac{\sum_i \frac{k_i x_i}{R_i^4}\left(1 - \frac{k_i NAE}{4\pi n_i R_i^2}\right)}{\sum_i \frac{k_i}{R_i^4}\left(1 - \frac{k_i NAE}{4\pi n_i R_i^2}\right)}, \tag{6}$$

etc. Since R_i is a function of $\mathbf{x}$, eqs. (4) and (6) are not closed-form solutions. However, we could start with an initial hypothesis as to (x, y, z), perhaps at the center of the detector, or based on a quick estimate such as $x = \sum_i n_i \mathbf{x}_i / \sum_i n_i$, and iterate for E and $\mathbf{x}$. Clearly, Monte Carlo studies are be needed to validate this approach.

This model can be augmented in various ways.

If we suppose that the photoelectrons are attenuated with distance R_i according to $e^{-R_i/\lambda}$, for a known attenuation length λ, the χ^2 would be modified to read,

$$\chi^2 = \sum_{i=1}^{192} \frac{\left(n_i - \frac{k_i NAE e^{-R_i/\lambda}}{4\pi R_i^2}\right)^2}{n_i}. \tag{7}$$

Reflectors at height z_+ and z_- with reflectivity ϵ could be accounted for by introducing,

$$R_{i\pm}^2 = (x - x_i)^2 + (y - y_i)^2 + (2z_\pm - z - z_i)^2, \tag{8}$$

and,

$$\chi^2 = \sum_{i=1}^{192} \frac{\left[n_i - k_i E\left(\frac{e^{-R_i/\lambda}}{R_i^2} + \epsilon\frac{e^{-R_{i+}/\lambda}}{R_{i+}^2} + \epsilon\frac{e^{-R_{i-}/\lambda}}{R_{i-}^2}\right)\right]^2}{n_i}. \tag{9}$$

The possibility of multiple reflections could be included by defining additional distances R_{i+-}, R_{i-+}, R_{i+-+}, *etc.*

The model (1) assumes that the solid angle of the photocathode depends only on the distance R_i and not on the direction. However, for a give R_i the solid angle actually varies by a (calculable) factor of ≈ 5 with direction, being smallest when the light travels nearly vertically in the detector [1]. Denoting this "obliquity factor" by $\alpha(\cos\theta_i)$, and noting that distance R_i is not necessarily large compared to the radius $r_1 = 9.5$ cm of the edge of the photocathode, eq. (1) could be modified as [1],

$$n_i = \frac{k_i N E\, e^{-R_i/\lambda}}{2} \left(1 - \frac{R_i}{\sqrt{R_i^2 + \alpha r_1^2}}\right) + \cdots. \tag{10}$$

As an example, consider the case that we include single reflections and an obliquity factor. Then,

$$n_i = \frac{k_i N E}{2} \sum_{j=1}^{3} \epsilon_j\, e^{-R_{ij}/\lambda} \left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2 + \alpha_{ij} r_1^2}}\right), \tag{11}$$

where the reflectivity ϵ_j is 1 for $j = 1$ and less for $j = 2$ and 3 which correspond to light having a single bounce at the top or bottom reflectors, $R_{i1} = R_i$, $R_{i2} = R_{i+}$, $R_{i3} = R_{i-}$, $\alpha_{ij} = \alpha(\cos\theta_{ij})$,

$$\cos\theta_{ij} = \frac{e_{ix}(x - x_i) + e_{iy}(y - y_i)}{R_{ij}}, \tag{12}$$

with $(e_{ix}, e_{iy}, 0)$ being a unit vector along the axis of PMT i. Following [1], the obliquity factor can be approximated as,

$$\alpha(\cos\theta) = \sum_{m=0}^{3} a_m \cos^m\theta, \tag{13}$$

where,

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 0, \qquad a_3 = 0 \qquad (\cos\theta > 0.69), \tag{14}$$

$$a_0 = 0.19, \qquad a_1 = 0, \qquad a_2 = 1.71, \qquad a_3 = -0.95 \qquad (\cos\theta < 0.69). \tag{15}$$

A cruder approximation is simply $\alpha = \cos\theta$, as would hold if the face of the PMT were flat.

Minimizing the χ^2 based on eq. (11) leads to,

$$\begin{aligned} 0 &= -\frac{1}{N}\frac{\partial\chi^2}{\partial E} \qquad (16) \\ &= \sum_i k_i \sum_{j=1}^{3} \epsilon_j\, e^{-R_{ij}/\lambda}\left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2 + \alpha_{ij} r_1^2}}\right)\left[1 - \frac{k_i N E}{2 n_i}\sum_{k=1}^{3}\epsilon_k\, e^{-R_{ik}/\lambda}\left(1 - \frac{R_{ik}}{\sqrt{R_{ik}^2 + \alpha_{ik} r_1^2}}\right)\right], \end{aligned}$$

$$E = \frac{\frac{2}{N}\sum_i k_i \sum_{j=1}^{3}\epsilon_j\, e^{-R_{ij}/\lambda}\left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2+\alpha_{ij} r_1^2}}\right)}{\sum_i \frac{k_i^2}{n_i}\left[\sum_{j=1}^{3}\epsilon_j\, e^{-R_{ij}/\lambda}\left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2+\alpha_{ij} r_1^2}}\right)\right]^2}, \tag{17}$$

$$\begin{aligned} 0 = \frac{1}{NE}\frac{\partial\chi^2}{\partial x} &= \sum_i k_i \sum_{j=1}^{3}\epsilon_j e^{-R_{ij}/\lambda}\left\{(x - x_i)\left[\frac{1}{\lambda R_{ij}}\left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2 + \alpha_{ij} r_1^2}}\right) + \frac{\alpha_{ij} r_1^2}{R_{ij}(R_{ij}^2 + \alpha_{ij} r_1^2)^{3/2}}\right]\right. \\ &\qquad \left. + \frac{\alpha'_{ij} r_1^2}{2(R_{ij}^2 + \alpha_{ij} r_1^2)^{3/2}}\left(\frac{\cos\theta_{ij}(x - x_i)}{R_{ij}} - e_{ix}\right)\right\} \\ &\qquad \left[1 - \frac{k_i N E}{2 n_i}\sum_{k=1}^{3}\epsilon_k\, e^{-R_{ik}/\lambda}\left(1 - \frac{R_{ik}}{\sqrt{R_{ik}^2 + \alpha_{ik} r_1^2}}\right)\right], \end{aligned} \tag{18}$$

where $\alpha' = d\alpha/d\cos\theta = \sum_{m=1}^{3} m\, a_m \cos^{m-1}\theta$. Hence, we can write,

$$x = \frac{A}{B}\,, \tag{19}$$

where,

$$\begin{aligned} A &= \sum_i k_i \sum_{j=1}^{3} \epsilon_j e^{-R_{ij}/\lambda} \left\{ x_i \left[\frac{1}{\lambda R_{ij}} \left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2 + \alpha_{ij} r_1^2}} \right) + \frac{[\alpha_{ij} + (\alpha'_{ij}/2)\cos\theta_{ij}] r_1^2}{R_{ij}(R_{ij}^2 + \alpha_{ij} r_1^2)^{3/2}} \right] \right. \\ & \left. - \frac{e_{ix}\alpha'_{ij} r_1^2}{2(R_{ij}^2 + \alpha_{ij} r_1^2)^{3/2}} \right\} \left[1 - \frac{k_i N E}{2 n_i} \sum_{k=1}^{3} \epsilon_k\, e^{-R_{ik}/\lambda} \left(1 - \frac{R_{ik}}{\sqrt{R_{ik}^2 + \alpha_{ik} r_1^2}} \right) \right], \end{aligned} \tag{20}$$

and,

$$\begin{aligned} B &= \sum_i k_i \sum_{j=1}^{3} \epsilon_j e^{-R_{ij}/\lambda} \left[\frac{1}{\lambda R_{ij}} \left(1 - \frac{R_{ij}}{\sqrt{R_{ij}^2 + \alpha_{ij} r_1^2}} \right) + \frac{[\alpha_{ij} + (\alpha'_{ij}/2)\cos\theta_{ij}] r_1^2}{R_{ij}(R_{ij}^2 + \alpha_{ij} r_1^2)^{3/2}} \right] \\ & \left[1 - \frac{k_i N E}{2 n_i} \sum_{k=1}^{3} \epsilon_k\, e^{-R_{ik}/\lambda} \left(1 - \frac{R_{ik}}{\sqrt{R_{ik}^2 + \alpha_{ik} r_1^2}} \right) \right]. \end{aligned} \tag{21}$$

Similar expressions hold for y, but the z-dependence in R_{ij} and $\cos\theta_{ij}$ leads to the substitutions $e_{ix} \to 0$, and $x_i \to 2z_\pm - z_i$ (rather than $x_i \to z_i$) for $j, k = 2, 3$ in the expressions for A and B.

For a maximum-likelihood approach to this issue, see [2]. An alternative to the analytic approach considered here is to use a Monte Carlo simulation of the detector, and to fit the results for the number of photoelectrons to some form $n_i = E f(x, y, z, x_i, y_i, z_i)$ whose derivatives with respect to x, y and z are reasonably simple. Then, an iterative χ^2 method could be constructed similar to that given in this note. A step in this direction has been made in [3].

Comments:

I wrote a simplified Monte Carlo event generator that assumes the faces of the PMTs are flat, and that there are no reflections. Using the actual E and $\mathbf{x}$ for the (point) source of scintillation, the reconstruction of energy E according to eq. (17) is reasonable, but the reconstruction of source position $\mathbf{x}$ according to eqs. (19)-(21) is poor.

I note that the form of eq. (20) is peculiar in that it weights PMT coordinate x_i according to $[n_i(\text{meas}) - n_i(\text{pred})]/n_i(\text{meas})$, which gives more significance to PMTs in which the observed signal differs more from the expectation. This may be a hint that the iterative algorithm considered here has poor convergence. Instead, it may be more appropriate to find the best-fit E and $\mathbf{x}$ by numerical minimization of the χ^2 (or by numerical maximization of the closely related likelihood function).

References

[1] K.T. McDonald, *Effective Center and Obliquity Factor of R5912 Photomultiplier Tubes* (July 11, 2011), http://kirkmcd.princeton.edu/dayabay/pmt_center.pdf

[2] L. Wen *et al.*, *Maximum likelihood reconstruction of a detector with reflective panels*, Nucl. Instr. and Meth. **A629**, 296 (2011),
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[3] S. Ryland Ely, *A Deterministic Code for Calculating the AD Light Response* (Apr. 29, 2009), http://dayabay.ihep.ac.cn/DocDB/0035/003525/001/lightsims.pdf