A Maxwellian Perspective on Particle Acceleration

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The Newtonian View

A charge e of mass m in fields $\mathbf{E}$ and $\mathbf{B}$ feels the Lorentz force:

$$\mathbf{F} = \gamma m \mathbf{a} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \qquad \text{(Gaussian units).}$$

$\Rightarrow$ Both $\mathbf{E}$ and $\mathbf{B}$ change the particle's momentum, but only $\mathbf{E}$ can change it's energy.

$\Rightarrow$ Integrate equation of motion to find everything you want to know...

Why do we need another perspective?

- Integration sometimes difficult.
- Forces sometimes obscure (Čerenkov radiation....).
- Useful to have a cross check.
- Useful to have a method for order-of magnitude estimation.

Personal motivation: Paradoxes of laser acceleration.

A Maxwellian Perspective

The key feature of particle acceleration is energy transfer between a charged particle and an electromagnetic field.

In Maxwell's view, the electromagnetic field stores energy:

$$U_{\rm field} = \int \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} d\mathrm{Vol},$$

and momentum:

$$\mathbf{P}_{\rm field} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\mathrm{Vol},$$

and angular momentum:

$$\mathbf{L}_{\rm field} = \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c} d\mathrm{Vol}.$$

Conservation laws $\Rightarrow$ if a particle gains energy (or $\mathbf{P}$ or $\mathbf{L}$) from the field, the field must lose an equal amount of energy (or $\mathbf{P}$ or $\mathbf{L}$).

$$\Delta U_{\rm gained\ by\ particle} = -\Delta U_{\rm lost\ by\ field}.$$

$\Rightarrow$ Understand particle acceleration by analysis of loss of field energy.

Separation of the Fields

We separate the electromagnetic fields into two parts:

- The **external** (or applied) fields. We suppose the sources of these field are not perturbed by the accelerating charge:

$$\Rightarrow U_{\text{ext}} = \int \frac{\mathbf{E}_{\text{ext}}^2 + \mathbf{B}_{\text{ext}}^2}{8\pi} d\text{Vol} = \text{constant}.$$

- The fields of the accelerating **charge**. These are characterized as near zone (quasistatic, Coulombic), induction zone, and far zone (radiation).

$$U_{\text{charge}} = \int \frac{\mathbf{E}_{\text{charge}}^2 + \mathbf{B}_{\text{charge}}^2}{8\pi} d\text{Vol} \to \infty,$$

  as the classical radius of the charge goes to zero.

  We perform a "classical renormalization", and declare $U_{\text{particle}} = U_{\text{mechanical}} + U_{\text{charge}}$ to be γmc^2, and ignore the issue of changes in U_{charge}.

  This ignores the radiated energy, and hence also the radiation reaction force.

The Interference Term

The part of the electromagnetic energy that changes to balance the energy gain of an accelerating particle is therefore the interference term:

$$U_{\text{int}} = \int \frac{\mathbf{E}_{\text{ext}} \cdot \mathbf{E}_{\text{charge}} + \mathbf{B}_{\text{ext}} \cdot \mathbf{B}_{\text{charge}}}{4\pi} d\text{Vol} = -\Delta U_{\text{gain}}.$$

For a charged particle to gain energy from an external field, this interference term must be negative.

The interference term (and energy gain) will be zero if the polarization of the external fields is orthogonal to that of the fields of the charge.

In general, the interference term (and energy gain) will be zero unless the fields of the charge include a component at the same frequency as that of the external fields.

We can often evaluate the interference term after the charge has left the confined region of the external fields. Then the interference is due to the radiation fields of the charge that were trapped in the confined region.

“Spontaneous” and “Stimulated” Fields

The **spontaneous** fields of the charge are those that exist when the external fields are set to zero. These fields differ from those of an isolated charge due to the presence of walls or other media.

Important examples of nontrivial spontaneous fields are image fields, transition radiation and Čerenkov radiation.

The **stimulated** fields of the charge are the additional fields that arise due to the response of the charge to the external fields.

In the first approximation, the strength of the stimulated fields is proportional to the strength of the external fields.

"Linear" Acceleration and "Quadratic" Acceleration

In the view of most people, "particle acceleration" is an effect that is **linear** in the strength of the external fields.

But the interference energy,

$$U_{\rm int} = \int \frac{\mathbf{E}_{\rm ext} \cdot \mathbf{E}_{\rm charge} + \mathbf{B}_{\rm ext} \cdot \mathbf{B}_{\rm charge}}{4\pi} d{\rm Vol} = -\Delta U_{\rm gain}.$$

will be linear in the external fields only if the fields of the charge are independent of the external fields.

Hence **linear** acceleration depends on the **spontaneous** part of the fields of the accelerating charge!

The interference between the external fields and the **stimulated** fields of the charge leads to **quadratic** acceleration (energy gain proportional to the square of the external field strength) and is usually ignored.

The Maxwellian Perspective

The Maxwellian perspective on particle acceleration emphasizes the spontaneous field of the charge as it passes through the accelerating structure.

Detailed prescription: Calculate the particle's trajectory in the presence of the external fields. Then evaulate the spontaneous fields on that trajectory, but with external fields set to zero.

Impulse approximation: Evaluate the spontaneous fields for the particle's trajectory in the absence of the external fields.

The Lawson-Woodward "Theorem"

If there is no accelerating "structure", there is no **linear** acceleration.

More precisely, a charged particle cannot gain net energy that is proportional to the strength of electromagnetic fields with which it interacts in vacuum over a finite path length, if all other matter is so remote that the spontaneous fields of the charged particle are negiligble.

This is "obviously" true in the impulse approximation of the Maxwellian perspective.

We may be able to give a subtle counterexample when we go beyond the impulse approximation.

Acceleration in a Static Field

Newton/Lorentz: Static electric field $E_0\hat{\mathbf{z}}$ over over distance $L \Rightarrow$ $\Delta U_{\rm gain} = eE_0L$.

Maxwell: when the charge is distance z from from one of the electrodes that supports the field E_0, the "spontaneous" electric field $\mathbf{E_e}$ at $\mathbf{r}$ due to the charge at z includes the field of the image charge $-e$ located at $-z$.

$$\mathbf{E}_e(\mathbf{r}, z) = \frac{e\mathbf{r}_1}{r_1^3} - \frac{e\mathbf{r}_2}{r_2^3}.$$

$$U_{\rm int} = \int \frac{E_0\hat{\mathbf{z}} \cdot \mathbf{E_e}(\mathbf{r}, z)}{4\pi} d{\rm Vol} = -eE_0z.$$

When the particle has traversed a potential difference $V = E_0L$, it has gained energy eV and the field has lost the same energy.

Acceleration in a Resonant Cavity

Cavity length L along the z axis.

Cavity field amplitude $E_0\hat{\mathbf{z}}$, frequency ω, wavelength $\lambda \gg L$.

$\Rightarrow$ wave number k obeys $kL \ll 1$.

Entrance and exit apertures of radius $a \ll L \Rightarrow a \ll \lambda$.

Newton/Lorentz: $\Delta U_{\rm gain} \approx eE_0L$, if appropriate phasing.

Acceleration is linear in E_0.

Maxwell: What is the spontaneous radiation at frequency ω?

First, spontaneous radiation as electron passes through an aperture in a metallic plate.

Zero aperture radius $\Rightarrow$ **transition radiation**.

$\Rightarrow \omega_{\rm rad} \lesssim \omega_{\rm plasma}$.

Aperture of radius $a, \Rightarrow$ only that part of the transition-radiation spectrum associated with radii greater than a.

Weizsäcker-Williams Approximation

Electron suddenly emerges from the metal plate.

$\Rightarrow$ Image charges on plate accelerate and radiate a pulse to maintain $\mathbf{E}_{\|} = 0$ on plate.

Radiation spectrum is Fourier transform of electron's field at plate.

At radius a, radial electric field of electron is $E \approx \gamma e/a^2$.

Pulse on plate lasts for time $\approx a/\gamma c$.

$\Rightarrow$ Spectrum extends up to wave number $k_{\rm max} \approx \gamma/a$.

$$U_{\rm pulse} \approx E^2 {\rm Vol} \approx \frac{\gamma^2 e^2}{a^4} \times a^2 \times \frac{a}{\gamma} = \frac{\gamma e^2}{a}.$$

$$\Rightarrow \frac{dU}{dk} \approx \frac{U}{k_{\rm max}} \approx e^2. \qquad \text{(Transition Radiation)}$$

[Detailed theory gives additional factor of $4/\pi$.]

Independent of a, $\Rightarrow$ valid for $k \lesssim \gamma/a$ if aperture a.

Photon Spectrum

$$dn = \frac{dU}{\hbar\omega} \approx \frac{e^2}{\hbar c}\frac{d\hbar\omega}{\hbar\omega} = \alpha\frac{d\omega}{\omega},$$

where $\omega = kc$ and $\alpha = e^2/\hbar c$.

In transition radiation, α photons are emitted per surface per unit bandwidth.

An RF Cavity Has Two Walls

Radiation from the second wall is emitted in the "backward" direction.

Spectrum is still $dU/dk \approx e^2$, but $k_{\rm max} \approx 1/a$.

"Backward" radiation has 180° phase shift relative to "forward".

Time delay of moving electron $\Rightarrow$ phase lag:

$$\Delta\varphi = kL\left(\frac{1}{\beta} - \cos\theta\right) \approx kL\left(1 - \cos\theta + \frac{1}{2\gamma^2}\right),$$

for radiation at angle θ.

To interfere with external field $E_0\hat{\mathbf{z}}$, need $\theta \approx 90^\circ$ where $\mathbf{E}_{\rm spont} \parallel \hat{\mathbf{z}}$.

$$\Rightarrow \Delta\varphi \approx kL \ll 1,$$

$$\mathbf{E}_2 = -e^{i\Delta\varphi}\mathbf{E}_1 = -e^{ikL}\mathbf{E}_1.$$

$$\mathbf{E}_{1+2} = \mathbf{E}_1(1 - e^{ikL}) \approx -ikLE_1\hat{\mathbf{z}}.$$

$$dU_{1+2} \approx k^2L^2 dU_1 = e^2L^2k^2dk.$$

Interference with One Cavity Mode

This energy excites various cavity modes.

Cavity-mode number density is

$$dN \approx k^2 dk \mathrm{Vol}, \qquad \text{(Rayleigh-Jeans)}.$$

$\Rightarrow$ Energy $U_{\rm rad}$ and field $E_{\rm rad}$ radiated into one mode are

$$U_{\rm rad} \approx \frac{dU_{1+2}}{dN} \approx \frac{e^2 L^2}{\rm Vol} \approx \mathbf{E}_{\rm rad}^2 \mathrm{Vol}.$$

$$\Rightarrow \qquad \mathbf{E}_{\rm rad} \approx -i \frac{eL}{\rm Vol} \hat{\mathbf{z}}.$$

[Could have been guessed by dimensional analysis.]

$$\text{Finally,} \qquad U_{\rm int} \approx (\mathbf{E}_0 \cdot \mathbf{E}_{\rm rad}) \mathrm{Vol} \approx -e E_{0,z} L = -U_{\rm gain},$$

for a suitable choice of phase of $\mathbf{E_0}$.

Acceleration by a Plane Wave?

"Practical" plane wave = far zone of a spherical wave.

Suppose amplitude rises slowly to plateau at E_0, then falls slowly.

Can an electron have nozero energy gain after passage of this wave?

NO, as first shown by di Francia and by Kibble in 1964.

We will give a Maxwellian argument.

But first, a disgression...

Momentum Balance for an Electron in a Plane Wave

A plane wave carries momentum along its wave vector, but shakes an electron transverse to the wave vector.

Analyze in frame in which electron is at rest on average.

$$\mathbf{E}_{\rm ext} = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \qquad \Rightarrow \qquad \mathbf{P}_{\rm mech} = m\dot{\mathbf{x}} = \frac{e}{\omega}\hat{\mathbf{x}} E_0 \sin\omega t,$$

which is linear in E_0.

[Ignoring the effect of $e\mathbf{v}/c \times \mathbf{B}_{\rm ext}$, which is quadratic in E_0.]

How is transverse momentum conserved?

Maxwell: Consider the field momentum.

In particular, consider the interference term involving the external fields and the fields of the electron.

The Interference Term $\mathbf{P}_{\text{ext,static}}$

$\mathbf{E}_e = \mathbf{E}_{\text{static}} + \mathbf{E}_{\text{osc}}$, but $\mathbf{B}_e = \mathbf{B}_{\text{osc}}$ only.

$\mathbf{E}_{\text{osc}}$ and $\mathbf{B}_{\text{osc}} \propto E_0$, since they are due to the motion of the electron caused by E_0.

Hence any interference term involving E_{ext} and E_{osc} will be quadratic in E_{ext} and cannot account for the transverse momentum.

$$\text{Only term left is} \qquad \mathbf{P}_{\text{ext,static}} = \int_V \frac{\mathbf{E}_{\text{static}} \times \mathbf{B}_{\text{ext}}}{4\pi c}.$$

$$\mathbf{E}_{\text{static}} = \frac{e}{r^3}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}), \qquad \mathbf{B}_{\text{ext}} = \hat{\mathbf{y}} E_0 \cos(kz - \omega t).$$

$$\begin{aligned}
\mathbf{P}_{\text{ext,static}} &= \int_V \frac{eE_0}{4\pi c r^3}\{-\hat{\mathbf{x}} z \cos(kz - \omega t) + \hat{\mathbf{z}} x \cos(kz - \omega t)\} \\
&= -\frac{e}{4\pi c}\hat{\mathbf{x}} E_x \sin\omega t \int_V \frac{z \sin kz}{r^3} = -\frac{e}{\omega}\hat{\mathbf{x}} E_x \sin\omega t \\
&= -\mathbf{P}_{\text{mech}}.
\end{aligned}$$

[Independent of any hypothesis as to the size of a classical electron.]

The Interference Field Energy

$$U_{\text{total}} = \int_V \frac{(\mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{static}} + \mathbf{E}_{\text{osc}})^2 + (\mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{osc}})^2}{8\pi}.$$

$$U_{\text{int}} = U_{\text{ext,static}} + U_{\text{ext,osc}}.$$

$$U_{\text{ext,static}} = \int_V \frac{\mathbf{E}_{\text{ext}} \cdot \mathbf{E}_{\text{static}}}{4\pi} \Rightarrow \langle U_{\text{wave,static}} \rangle_t = 0.$$

$$U_{\text{ext,osc}} = \int_V \frac{\mathbf{E}_{\text{wave}} \cdot \mathbf{E}_{\text{osc}} + \mathbf{B}_{\text{wave}} \cdot \mathbf{B}_{\text{osc}}}{4\pi}.$$

$$\langle U_{\text{ext,osc}} \rangle_t = -\frac{2e^2 E_0^2}{3m\omega^2} = -\frac{4}{3}\eta^2 mc^2,$$

$$\eta = \frac{e\sqrt{\langle A_\mu A^\mu \rangle_t}}{mc^2} = \frac{eE_{0,\text{rms}}}{m\omega c} = \frac{eE_{0,\text{rms}}\lambda}{mc^2},$$

where A is the four-vector potential.

The Mass Shift

Because of its motion in the external field, the electron undergoes a relativistic mass increase.

The time-average mass is $\overline{m} = m\sqrt{1+\eta^2}$ (Kibble, 1965).

For a weak field, $\eta \ll 1$ the mass increase is $\Delta m \approx \eta^2 m/2$.

Maxwell: This mass increase should be compensated by a decrease in field energy.

We find $\langle U_{\rm int}\rangle = -8\Delta mc^2/3$.

This appears to be a variant of the factor of 4/3 that appears in some classical analyses of the electromagnetic energy and momentum of a charged particle.

Net Energy Transfer?

The integrand† of $\langle U_{\rm int}\rangle$ is significant only for distances of order λ from the charged particle.

Hence, $\langle U_{\rm int}\rangle$ returns to zero once the wave has passed the particle by.

$\Rightarrow$ No net energy transfer of a charged particle by a plane wave.

†The integrand is

$$\frac{k^2 z}{r^2}\sin kz\ \sin kr + \frac{kz}{r^3}\sin kz\ \cos kr$$
$$+\left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\cos kz\ \sin kr$$
$$+\left(\frac{k^2}{r} - \frac{k^2 y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\cos kz\ \cos kr.$$

Temporary Acceleration by a Plane Wave

If the electron had been at rest before the arrival of the plane wave, it would take on a drift velocity,

$$v_z = (\eta^2/2)/(1 + \eta^2/2),$$

once inside the wave.

The corresponding energy of the electron inside the wave is $mc^2(1 + \eta^2/2)$.

For $\eta \gg 1$, the longitudinal motion of the electron inside the wave is highly relativistic.

Could an electron interact with a real laser pulse (limited in extent in all of x, y, z and t) so as to retain some or all of the energy it gains while inside the wave?

This is the tantalizing prospect of **laser acceleration**.

But remember: no spontanteous radiation from a charge in vacuum far from all $\Rightarrow$ laser acceleration depends of interference of **stimulated** radiation.

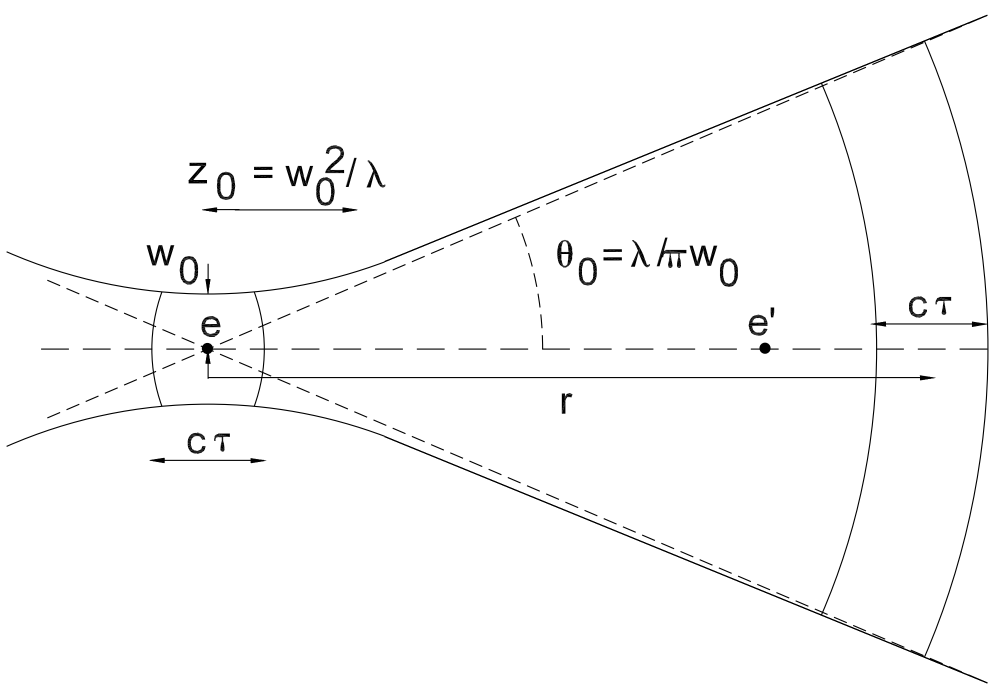
Maxwell and the Laser

Physics textbooks don't discuss solutions to Maxwell's equations that correspond to laser beams.

Laser textbooks don't discuss solutions to Maxwell's equations that are accurate enough to understand the physics of laser acceleration.

Try: $E_x = E_0 e^{-\rho^2/w_0^2} e^{i(kz-\omega t)}$, where w_0 = laser **waist**.

Doesn't contain **diffraction**: need $w = w(z)$ where $w \to z\theta_0$ at large z, and $\theta_0 = \lambda/\pi w_0 = 2/kw_0$ = **diffraction angle**.



Gaussian Laser Beams, I

Introduce **Rayleigh range**: $z_0 = w_0/\theta_0 = \pi w_0^2/\lambda = 2/k\theta_0^2$.

Then $w^2(z) = w_0^2(1+\zeta^2)$ with $\zeta = z/z_0$.

Now, $E_x = E_0 e^{-\rho^2/w_0^2(1+\zeta^2)} e^{i(kz-\omega t)}$.

Doesn't satisfy **energy conservation**: Near focus, $U \approx E_0^2 w_0^2 c\tau$, while in far field $U \approx E^2(z) w_0^2(1+\zeta^2) c\tau$.

$\Rightarrow E(z) = E_0/\sqrt{1+\zeta^2}$.

$$\text{Now,} \qquad E_x = \frac{E_0 e^{-\rho^2/w_0^2(1+\zeta^2)}}{\sqrt{1+\zeta^2}} e^{i(kz-\omega t)}.$$

Wave fronts in far zone aren't spherical:

$r = \sqrt{\rho^2+z^2} \approx z(1+\rho^2/2z^2), \quad \Rightarrow$

$$e^{ikr} \approx e^{ikz+ik\rho^2/2z} = e^{ikz} e^{ikw_0^2\rho^2/2w_0^2 z} = e^{ikz} e^{i\rho^2/w_0^2\zeta} \approx e^{ikz} e^{i\zeta\rho^2/w_0^2(1+\zeta^2)}$$

$$\text{Now,} \qquad E_x = \frac{E_0 e^{-(1-i\zeta)\rho^2/w_0^2(1+\zeta^2)}}{\sqrt{1+\zeta^2}} e^{i(kz-\omega t)}.$$

Gaussian Laser Beams, II

Introduce $$f = \frac{1-i\zeta}{1+\zeta^2} = \frac{1}{1+i\zeta} = \frac{e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}}.$$

Now, $$E_x = \frac{E_0 e^{-f\rho^2/w_0^2}}{\sqrt{1+\zeta^2}} e^{i(kz-\omega t)}.$$

Doesn't contain the **Guoy phase shift** (1895).

Kirchhoff scalar diffraction theory

$\Rightarrow \psi(0,z)$ related to $\psi(\rho,0) = \psi_0 \exp(-\rho^2/w_0^2)$ via

$$\begin{aligned}
\psi(0,z) &= \frac{k}{2\pi i}\int_{\mathrm{A}} \psi(\rho,0)\frac{e^{ikr}}{r} \approx \frac{k\psi_0}{2i}\frac{e^{ikz}}{z}\int_0^\infty d\rho^2 e^{-\rho^2/w_0^2} \\
&= -i\frac{kw_0^2}{2}\psi_0\frac{e^{ikz}}{z} = -i\frac{z_0}{z}\psi_0 e^{ikz} = -\frac{i}{\zeta}\psi_0 e^{ikz} \\
&\approx f\psi_0 e^{ikz} = \frac{\psi_0 e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}} e^{ikz}.
\end{aligned}$$

The **Guoy phase shift** is $-\tan^{-1}\zeta$, which is

$-\pi/2$ at $z = -\infty$,

0 at $z = 0$,

$\pi/2$ at $z = +\infty$.

Gaussian Laser Beams, III

Now, $\quad E_x = E_0 f e^{-f\rho^2/w_0^2} e^{i(kz-\omega t)}.$

This satisfies the **paraxial wave equation**:

$$\nabla_\perp^2 \psi + 4i\frac{\partial \psi}{\partial \zeta} = 0,$$

$$\text{where} \qquad \nabla_\perp^2 = \frac{w_0^2 \partial^2}{\partial x^2} + \frac{w_0^2 \partial^2}{\partial y^2}.$$

The paraxial wave equation is a first approximation to

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2},$$

which supposes that the diffraction angle, θ_0, is small.

But, doesn't satisfy $\nabla \cdot \mathbf{E} = 0$.

$\partial E_x/\partial x \neq 0, \quad \Rightarrow \quad \partial E_z/\partial z \neq 0, \quad \Rightarrow \quad E_z \neq 0.$

$\Rightarrow$ A Gaussian laser beam must have a **longitudinal electric field**.

Lowest Linearly Polarized Gaussian Laser Mode

Better to use our Gaussian wave for the **vector potential, A**.

$$A_x = E_0 f e^{-f\rho^2/w_0^2} e^{i(kz-\omega t)}.$$

Lorentz gauge $\Rightarrow$ the scalar potential ϕ obeys

$$\frac{\partial \phi}{\partial t} = -c\nabla \cdot \mathbf{A}, \qquad \Rightarrow \qquad \phi = -\frac{i}{k}\nabla \cdot \mathbf{A}.$$

The electric and magnetic fields can now be deduced from the approximate vector potential via

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k}\nabla(\nabla\cdot\mathbf{A}) + ik\mathbf{A}, \qquad \text{and} \qquad \mathbf{B} = \nabla\times\mathbf{A},$$

The results to order θ_0 are, after dividing out a factor of ik:

$$E_x = \psi_0 e^{i\varphi}, \qquad E_y = 0, \qquad E_z = \frac{i\theta_0}{2}\frac{\partial\psi_0}{\partial\xi}e^{i\varphi} = -i\theta_0 f\xi E_x,$$

$$B_x = 0, \qquad B_y = E_x, \qquad B_z = \frac{i\theta_0}{2}\frac{\partial\psi_0}{\partial\upsilon}e^{i\varphi} = -i\theta_0 f\upsilon E_x.$$

where $\varphi = kz - \omega t$, $\xi = x/w_0$ and $\upsilon = y/w_0$.

These expressions satisfy $\nabla\cdot\mathbf{E} = 0 = \nabla\cdot\mathbf{B}$ to order θ_0^2.

Lowest Radially Polarized Gaussian Laser Mode

We could take our vector potential to be A_y or A_z as well.

If A_y, get electric field linearly polarized along y.

For $A_z = E_0 f e^{-f\rho^2/w_0^2} e^{i\varphi}$, use cylindrical coords:

$$E_\rho = \varrho F_0, \qquad E_\phi = 0, \qquad E_z = i\theta_0(1 - f^2\varrho^2)F_0,$$

$$F_0 = E_0 \frac{e^{-\varrho^2/(1+\zeta^2)}}{1+\zeta^2} e^{i(\varphi - 2\tan^{-1}\zeta)},$$

where $\varrho = \rho/w_0$. The Guoy phase shift for this mode is twice that for the linearly polarized mode.

These Gaussian modes are only the lowest of sets of modes, which are the paraxial approximations to oblate spheroidal wave solutions to Maxwell's equations.

[See chap. 21 of Abramowitz and Stegun.]

Acceleration by an Axicon Laser Mode

A plausible configuration for laser acceleration has the wave vector of the laser at an angle to the velocity of the electron.

Then $\mathbf{F}\cdot\mathbf{v} = e\mathbf{E}\cdot\mathbf{v} \neq 0$, and net energy transfer may be possible.

A symmetrized configuration with this property is an **axicon focus** beam with radial electric-field polarization.

The simplest cylindrical Gaussian laser mode with radial polarization is the (0,0) mode:

$$E_\rho = \varrho F_0, \qquad E_\phi = 0, \qquad E_z = i\theta_0(1 - f^2\varrho^2)F_0,$$

$$F_0 = E_0 \frac{e^{-\varrho^2/(1+\zeta^2)}}{1+\zeta^2} e^{i(\varphi - 2\tan^{-1}\zeta)},$$

$$f = \frac{1}{1+i\zeta}, \qquad \varrho = \frac{\rho}{w_0}, \qquad \zeta = \frac{z}{z_0} \qquad \text{and} \qquad \varphi = kz - \omega t.$$

The Guoy Phase Shift

The transverse field, E_ρ, has phase shift $-2\tan^{-1}(\infty) = -180°$ between the laser focus and the far field.

The axial field, E_z, is 90° out of phase with the transverse field.

Hence, the axial field at the focus is 90° out of phase with the far transverse field.

No Vacuum Laser Acceleration by an Axicon Beam

Maxwell: Energy gain requires nozero interference between the external (laser) field and the radiation of the electron.

Calculate in far field where laser pulse no longer overlaps the electron (assumed to be relativistic).

There, the axicon laser field has transverse radial polarization.

The radiation fields due from the acceleration of the electron at the focus by the longitudinal electric field do have transverse radial polarization in the far zone.

But the laser fields are 90° out of phase with the radiated fields in the far zone,
⇒ Vanishing interference energy,
⇒ No net energy gain.

[Variant: Phase slippage between electron and laser pulse nullifies any energy transfer over a long path.]

But Laser Cavity Acceleration is Possible

Limit path length of interaction of electron and axicon beam to $2L$ via mirrors which define a **laser cavity**.

$\Rightarrow$ Transition radiation from (relativistic) electron at first mirror can interfere with the laser beam, and net energy transfer is possible.

[Variant: Short path length keeps phase slippage small. Cavity length should be less than the "formation length":

$$L_{\rm formation} = \frac{2\lambda}{1-\beta} \approx \gamma^2 \lambda.]$$

Overlap of Laser Pulse with Transition Radiation

Must consider that part of transition-radiation spectrum that overlaps the laser frequency.

$$\text{Transition radiation :} \qquad dU_1 \approx e^2 dk = e^2 d\omega/c.$$

Laser pulse length $= \tau, \quad \Rightarrow \Delta\omega = 1/\tau$.

$\Rightarrow U_1 \approx e^2/(c\tau)$ radiated into laser bandwidth.

The pulse length of this transtion radiation is $\approx \tau$ also.

This radiation must also overlap the laser pulse in space.

Characteristic angle of radiation is $1/\gamma$.

$\Rightarrow$ Radiation extends over radius L/γ at center of cavity.

$\Rightarrow$ Field volume $\approx c\tau(L/\gamma)^2$.

$$\Rightarrow \qquad U_1 \approx c\tau \frac{L^2}{\gamma^2} E_1^2 \approx \frac{e^2}{c\tau}, \qquad \text{and} \qquad E_1 \approx \frac{\gamma e}{c\tau L}.$$

Radius of interference volume $= \min\{w_0, L/\gamma\}$.

($w_0 =$ laser waist)

Short Cavity ($L < \gamma w_0$)

Interference radius is L/γ.

$$U_{\text{int}} \approx -E_0 E_1 c\tau \frac{L^2}{\gamma^2} \approx -eE_0 \frac{L}{\gamma},$$

for suitable choice of phase.

Match focus of laser to Lorentz factor of electron:

$\Rightarrow$ Diffraction angle $\theta_0 = w_0/z_0 = 1/\gamma$,

$$\Delta U_e = -U_{\text{int}} \approx eE_0 w_0 \frac{L}{z_0} \approx eE_{0,z} L.$$

Energy gain increases linearly with cavity length.

But for $L < z_0$ the cavity is "long" and the transition radiation doesn't overlap completely with the laser waist.

Long Cavity ($L > \gamma w_0 \approx z_0$)

Interference radius is w_0.

$$U_{\text{int}} \approx -E_0 E_1 c\tau w_0^2 \approx -eE_0 \frac{\gamma w_0^2}{L},$$

for suitable choice of phase.

Again, matching $\Rightarrow \gamma w_0 \approx z_0$,

$$\Delta U_e = -U_{\text{int}} \approx eE_0 w_0 \frac{z_0}{L} \approx eE_{0,z} z_0 \frac{z_0}{L}.$$

The energy gain is suppressed for cavity lengths greater than the Rayleigh range.

Direct Calculation of Laser Cavity Acceleration

Consider $E_z(0,z)$ of a long radially polarized (0,0) laser mode interacting with a relativistic electron over path length $2L$:

$$\begin{aligned}\Delta U_e(L) &= e\int_{-L}^{L} E_z(0,z)dz = eE_{0,z}\int_{-L}^{L} dz\frac{z_0^2}{z^2+z_0^2}\cos[2\tan^{-1}(z/z_0)] \\ &= 2eE_{0,z}z_0\frac{Lz_0}{L^2+z_0^2},\end{aligned}$$

for an optimal choice of phase.

[⇒ Energy gain vanishes for large L.]

The maximum energy gain is attained at the matching condition, $L = z_0$ (Rayleigh range):

$$\Delta U_{e,\max} = eE_{0,z}z_0 = eE_0\theta_0 z_0 = eE_0 w_0.$$

Since $w_0 = z_0\theta_0 \approx L/\gamma$, the energy gain is only

$$\Delta U_{e,\max} \approx \frac{eE_0 L}{\gamma},$$

and laser cavity acceleration is less and less effective at higher energies.

Aperture Restriction

In practice the mirrors of a laser cavity would have apertures for the electron beam.

But if aperture is too large, spectrum of aperture radiation won't overlap the laser frequency,

$\Rightarrow$ Interference term vanishes,

$\Rightarrow$ Acceleration vanishes.

[Variant: If the apertures are too large, the field on axis is perturbed too much and can't accelerate electrons.]

Recall: Spectrum cuts off at $k \approx \gamma/a$,

$$\Rightarrow \quad \text{Need} \qquad a \lesssim \gamma\lambda.$$

A serious restriction for low γ.

Inverse Čerenkov Acceleration, I

If the laser cavity is filled with a gas of index $n \gtrsim 1$, the "spontaneous" radiation of an electron includes Čerenkov radiation, at angle

$$\theta_C = \cos^{-1}\frac{1}{n\beta} \approx \sqrt{(n-1)^2 - \frac{1}{\gamma^2}}.$$

Čerenkov radiation is radially polarized, and can interfere with an axicon laser mode whose diffraction angle is $\theta_0 \approx \theta_C$.

A suitable choice of index gives $\theta_C \approx 1/\gamma \approx \theta_0$.

The laser cavity should not be longer than the formation length, to avoid phase slippage: $kL \approx \gamma^2 \approx 1/\theta_0^2$.

The photon-number spectrum of Čerenkov radiation is

$$dN_C \approx \alpha\theta_C^2 \Delta\omega L/c,$$

so for a laser pulse of length τ, the energy spectrum is

$$dU_C \approx \frac{e^2\theta_0^2 kL}{c\tau} \approx \frac{e^2}{c\tau} \approx E_C^2 (L\theta_0)^2 c\tau, \qquad \Rightarrow \qquad E_C \approx \frac{e}{L\theta_0 c\tau}.$$

Inverse Čerenkov Acceleration, II

The interference energy with the laser beam is

$$-U_{\rm int} \approx E_0 E_C (L\theta_0)^2 c\tau \approx eE_0 L\theta_0 \approx \frac{eE_0 L}{\gamma} = U_{\rm gain},$$

for a suitable choice of phase.

There is no real advantage of inverse Čerenkov acceleration over laser cavity acceleration via "inverse transition radiation".

Particle interactions in the gas are a negative.

Vacuum Acceleration of an Electron Initially at Rest Near the Focus of a Weak Linearly Polarized Laser

"Vacuum acceleration" $\Rightarrow$ no nearby mirrors,

$\Rightarrow$ No "spontaneous" radiation,

$\Rightarrow$ Energy gain quadratic in laser field strength.

Linearly polarized laser:

$$E_x = E_0 f e^{-f\varrho^2} e^{i\varphi}, \qquad E_y = 0, \qquad E_z = \frac{i\theta_0}{2}\frac{\partial\psi_0}{\partial\xi} e^{i\varphi} = -i\theta_0 f \xi E_x,$$

$$f = \frac{1}{1+i\zeta}, \qquad \varrho = \frac{\rho}{w_0}, \qquad \zeta = \frac{z}{z_0} \qquad \text{and} \qquad \varphi = kz - \omega t.$$

Radiation due to E_z near focus is radially polarized in far zone,

$\Rightarrow$ No net interference with linearly polarized laser in far zone,

$\Rightarrow$ No net energy gain from E_z.

Dipole Radiation

Consider dipole radiation of a nonrelativistic electron at (average) position $\mathbf{x} = (0, 0, z)$, with $z \lesssim z_0$, in response to $\mathbf{E}_x$.

The dipole moment $\mathbf{d}$ obeys:

$$\ddot{\mathbf{d}} = e\ddot{\mathbf{x}} \approx \frac{e^2}{m}\mathbf{E}_x(\mathbf{x}) = c^2 r_e \mathbf{E}_x(\mathbf{x}),$$

$where r_e = e^2/mc^2$ is the classical electron radius.

The radiation field $\mathbf{E}_{\rm rad}(\mathbf{r}, t)$ of the electron at distance $r \gg z_0$ is obtained from $\ddot{\mathbf{d}}$ when evaluated at the retarded time,

$t' = t - |\mathbf{r} - \mathbf{x}|/c \approx t - r/c + z/c.$

$$\begin{aligned} \mathbf{E}_{\rm rad}(\mathbf{r}, t) \ &= \ \frac{(\ddot{\mathbf{d}}(t') \times \hat{\mathbf{k}}) \times \hat{\mathbf{k}}}{c^2 r} \approx -\frac{\ddot{\mathbf{d}}(t')}{c^2 r} \\ &\approx \ -\frac{r_e}{r} E_0 e^{-i\varphi_{\rm rad}} \widehat{\mathbf{E}}_0 \approx -\frac{e\eta}{r\lambda} e^{-i\varphi_{\rm rad}} \widehat{\mathbf{E}}_0, \end{aligned}$$

for $\theta \lesssim \theta_0$, where $\varphi_{\rm rad}(z) \approx z/z_0$ and $\eta = eE_{0,{\rm rms}}/m\omega c \ll 1$ (weak laser).

Energy Gain

The laser fields occupy volume $\approx (r\theta_0)^2 c\tau$ in the far zone, for a pulse of length $c\tau \lesssim z_0$.

$$\begin{aligned} U_{\rm int} &\approx E_0 E_{\rm rad} \cos(\pi/2 - \varphi_{\rm rad})(r\theta_0)^2 c\tau \approx -\theta_0^2 c\tau E_0^2 z_0 r_e \varphi_{\rm rad}(z) \\ &\approx -\eta^2 mc^2 \frac{c\tau}{\lambda}\frac{z}{z_0}. \end{aligned}$$