The Hawking-Unruh Temperature and Damping in a Linear Focusing Channel

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Abstract

The Hawking-Unruh effective temperature, $\frac{\hbar a^\star}{2\pi ck}$, due to quantum fluctuations in the radiation of an accelerated charged-particle beam can be used to show that transverse oscillations of the beam in a practical linear focusing channel damp to the quantum-mechanical limit. A comparison is made between this behavior and that of beams in a wiggler.

1 Introduction

Many of the effects of quantum fluctuations on the behavior of charged particles can be summarized concisely by an effective temperature first introduced in gravitational fields by Hawking [1], and applied to accelerated particles (with the neglect of gravity) by Unruh [2].

Hawking argued that the effect of the strong gravitational field of a black hole on the quantum fluctuations of the surrounding space is to cause the black hole to radiate with a temperature

$$T = \frac{\hbar g}{2\pi ck}, \tag{1}$$

where g is the acceleration due to gravity at the surface of the black hole, c is the speed of light, and k is Boltzmann's constant. Shortly thereafter, Unruh argued that an accelerated observer should become excited by quantum fluctuations to a temperature

$$T = \frac{\hbar a^\star}{2\pi ck}, \tag{2}$$

where $a^\star$ is the acceleration of the observer in its instantaneous rest frame.

In a series of papers, Bell and co-workers [3], have noted that electron storage rings provide a demonstration of the utility of the Hawking-Unruh temperature (2), with emphasis on the question of the incomplete polarization of the electrons due to quantum fluctuations of synchrotron radiation. The author has commented on how the Hawking-Unruh temperature can be used to characterize quickly the limits on damping of the phase volume of beams in electron storage rings [4], leading to well-known results of Sands [5].

2 Quantum Analysis of a Linear Focusing Channel

Recently, Chen, Huang and Ruth have discussed radiation damping in a linear focusing channel [6, 7, 8], finding that in such devices the beam can be damped to the quantum mechanical limit set by the uncertainty principle. I show here how this result follows very quickly from an application of the Hawking-Unruh temperature.

A linear focusing channel is a beam-transport system that confines the motion of a charged particle along a straight central ray via a potential that is quadratic in the transverse spatial coordinates. This potential can be characterized by a spring constant k, and hence the frequency ω of transverse oscillations (as observed in the laboratory frame) of a particle of mass m and Lorentz factor γ is

$$\omega = \sqrt{\frac{k}{\gamma m}}. \tag{3}$$

If the amplitude of the oscillation in transverse coordinate x is called x_0, then the amplitude a_0 of the corresponding transverse acceleration is

$$a_0 = x_0 \omega^2 = \frac{k x_0}{\gamma m}. \tag{4}$$

To apply the Hawking-Unruh temperature, we consider the motion in the instantaneous rest frame of the particle. Supposing the transverse oscillations are small, the instantaneous rest frame is very nearly the frame in which the particle has no longitudinal motion. Quantities measured in the instantaneous rest frame will by denoted with the superscript $\star$. Thus, in the instantaneous rest frame the amplitude of the transverse acceleration as measured is

$$a_0^\star = \gamma^2 a_0 = \frac{\gamma k x_0}{m}, \tag{5}$$

the frequency of the oscillation is

$$\omega^\star = \gamma\omega, \tag{6}$$

and hence the transverse spring constant of the focusing channel appears as

$$k^\star = m\omega^{\star 2} = \gamma k. \tag{7}$$

In the instantaneous rest frame, the charge particle finds itself in a bath of radiation of characteristic temperature given by eq. (2) with acceleration $a^\star$ given by eq. (5). This bath can be regarded as the effect of quantum fluctuations, which excite transverse oscillations (having two degrees of freedom) to characteristic energy $U^\star$ (as measured in the instantaneous rest frame) given by

$$U^\star = kT = \frac{\hbar a_0^\star}{2\pi c} = \frac{\hbar \gamma k x_0}{2\pi m c}. \tag{8}$$

The energy of transverse oscillation can also be written in terms of the (invariant) transverse amplitude x_0 as

$$U^\star = \frac{k^\star x_0^{\star 2}}{2} = \frac{\gamma k x_0^2}{2}. \tag{9}$$

Hence, the amplitude of excitation of the transverse oscillations is

$$x_0 = \frac{\hbar}{\pi m c} = \frac{\lambda\!\!\!^{-}_C}{\pi}, \tag{10}$$

where $\lambda\!\!\!^{-}_C$ is the (reduced) Compton wavelength of the particle.

The amplitude (10) must, however, be compared to the amplitude of the zero-point oscillations of the system, considered as a quantum oscillator:

$$x_{0,\text{zero point}} = \sqrt{\frac{\hbar}{\gamma m\omega}} = \sqrt{\frac{\lambda\!\!\!^{-}_C \lambda\!\!\!^{-}}{\gamma}}, \tag{11}$$

where $\lambda\!\!\!^{-} = c/\omega$ is the laboratory (reduced) wavelength of the transverse oscillation as measured along the beam axis. In practical laboratory devices, we will have $\lambda\!\!\!^{-} \gg \gamma\lambda\!\!\!^{-}_C$. Hence, the excitation of the transverse oscillations by fluctuations in the radiation of the oscillating charge, as are described by the Hawking-Unruh temperature, is negligible compared to the zero-point fluctuations of the transverse oscillations. In this sense, we can say along with Huang, Chen and Ruth that the radiation does not excite the transverse oscillations, and those oscillations will damp to the quantum-mechanical limit.

In futuristic devices, for which $\gamma > \lambda\!\!\!^{-}/\lambda\!\!\!^{-}_C$, *i.e.*, when

$$\gamma > \frac{mc^2}{k\lambda\!\!\!^{-}_C}, \tag{12}$$

quantum excitations of oscillations in a linear focusing channel would become important. When (12) holds, the transverse oscillations would be relativistic even when their amplitude is only a Compton wavelength. The strength of the transverse fields in the channel would then exceed the QED critical field strength (in the average rest frame),

$$E_{\text{crit}} = \frac{m^2c^3}{e\hbar} = 1.6 \times 10^{16}\ \text{V/cm} = 3.3 \times 10^{13}\ \text{Gauss}, \tag{13}$$

and the beam energy would be rapidly dissipated by pair creation.

Another way of viewing a practical linear focusing channel is that its Hawking-Unruh excitation energy, (8), is small compared to the zero-point energy, $\hbar\omega^\star/2 = \gamma\hbar\omega/2$ of transverse oscillations.

The quantum-mechanical limit for transverse motion can, of course, also be deduced from the uncertainty principle:

$$\sigma_x \sigma_{p_x} \gtrsim \hbar, \tag{14}$$

which leads to a minimum normalized emittance of

$$\epsilon_N = \frac{\sigma_x \sigma_{p_x}}{mc} \approx \lambda\!\!\!^{-}_C, \tag{15}$$

corresponding to geometric emittance of

$$\epsilon_x = \frac{\epsilon_N}{\gamma\beta_z} \approx \frac{\lambda\!\!\!^{-}_C}{\gamma}. \tag{16}$$

3 Semiclassical Analysis

In a quantum analysis of a linear focusing channel, we found that the transverse oscillations can damp to the limit set by the uncertainty principle. Hence, in a classical analysis we would expect the damping to be able to proceed until the transverse amplitude was zero.

Indeed, a simple analysis confirms this. Transform to the longitudinal rest frame, in which the particle's motion is purely transverse. The particle has nonzero kinetic energy in this frame, but its average momentum is zero. The radiation due to the transverse oscillation is reflection symmetric about the transverse plane in this frame, so the radiation carries away energy but not momentum. With time, all of the energy would be radiated away, and the particle would come to rest. The transverse oscillations will have damped to zero without affecting the longitudinal motion.

If we add the concept of photons to the preceding analysis, we can say that the radiated photons carry away momentum along the direction of emission, but the radiation pattern is symmetric, so the averaged radiated momentum is zero. Again, the radiation carries away energy, now in the form of photons.

Back in the lab frame, we view the photons as carrying away a small amount of longitudinal momentum on average, as a result of the Lorentz transformation of the energy radiated in the longitudinal rest frame. This momentum, however, is only that part of the particle's longitudinal momentum associated with its transverse oscillation; the longitudinal velocity of the particle is unaffected.

On average, the photons carry away no transverse momentum in the lab frame, and the average momentum of the radiated photons is therefore parallel to the beam axis in lab frame. However, there is no need to argue that the momentum of individual radiated photons is parallel to the beam axis, nor to imply that the matter of the focusing channel absorbs transverse momentum in a manner than affects the kinematics of the radiation process [7].

4 Comparison with a Wiggler

A comparison with the behavior of particle beams in a wiggler is instructive. Here the transverse confinement of the beam motion is provided by a series of alternating transverse magnetic fields. This has the notable effect that even if a particle enters the wiggle parallel to the beam axis, transverse oscillations will result whose amplitude is independent of the initial transverse coordinate.

In contrast, a particle that enters a linear focusing channel parallel to and along the axis undergoes no oscillation, no matter what is the particle's longitudinal momentum.

We thereby see that radiation damping cannot reduce the oscillations in a wiggler to zero unless the longitudinal momentum falls to zero also, since the wiggler continually re-excites transverse oscillations for any particle with nonzero kinetic energy.

Another difference between a wiggler and a linear focusing channel can be seen by going to the longitudinal rest frame. In the case of the wiggler, the alternating magnetic fields in the laboratory transform to fields that are very much like a plane wave propagating against the direction of the laboratory motion of the beam. The radiation induced by this effective plane wave is not symmetric with respect to the transverse plane, but results in a net kick of the particle into the backward direction.

Viewed in the lab frame, we find that along with the damping of their transverse oscillations, the particles' longitudinal momenta are significantly reduced. To maintain the initial longitudinal momentum, the beam must be reaccelerated. The momentum (and energy) added back into the beam then increases the amplitude of the transverse oscillations, and the damping cannot continue beyond some limit.

In contrast, in a linear focusing channel, the transverse damping proceeds without significant reduction in the longitudinal momentum of the particle, and the transverse oscillations can damp to the quantum limit without the need of adding energy back into the beam.

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