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# Large-sphere limit of the radar back-scattering coefficient

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SUMMARY

In the limit of large scatterer diameters, the Mie theory must converge to geometrical optics. Previous results of Mie-theory calculations of the radar back-scattering coefficient $\sigma_b$ have suggested that its large-sphere limit is the ordinary reflection coefficient $R$ of the scattering material. A proof that this is the asymptotic value of $\sigma_b$ is given.

## 1. Introduction

The scattering and absorption of plane electromagnetic waves by spheres whose diameters are neither very large nor very small compared to the incident wavelength are predictable on the Mie theory (Van de Hulst 1957). In radar practice, particular interest centres around just the small portion of the scattered radiation which is sent back upon the incident ray towards the radar set. The efficiency with which drops or ice-spheres scatter radar energy in this backward direction is expressed in terms of the back-scattering cross-section $\sigma$, defined as the area which, when multiplied by the incident power density $P_i$, gives the *total* power that would be radiated by an imaginary *isotropic* source radiating the same power density $P_s$ in the backward direction as does the actual scatterer. That is $\sigma P_i = 4\pi r^2 P_s$ if $r$ is the distance from radar set to scatterer (see Battan 1959, p. 25, for an alternate version of this intrinsically awkward definition). It is important here to note that the back-scattered power density $P_s$ entering into the definition is a flux density per unit *area* (not per unit solid angle) and also that $P_s$ is to be regarded as measured at the position of the radar set, whereas $P_i$ is measured at the scatterer. For many purposes it is more convenient to normalize $\sigma$ through division by the scattering sphere's geometric cross-section, yielding the dimensionless back-scattering coefficient $\sigma_b$. If $a$ is the radius of the scatterer we write

$$\sigma_b = \sigma/\pi a^2 = (4r^2 P_s)/(a^2 P_i) \quad . \quad . \quad . \quad . \quad (1)$$

The Mie theory permits calculation of $\sigma_b$ in terms of the optical properties of the scatterer and the dimensionless size parameter $\alpha = 2\pi a/\lambda$ where $\lambda$ is the wavelength of the radiation.

In a variety of back-scattering problems it is of interest to know whether $\sigma_b$ converges to any limit as $\alpha$ grows large; and in calculating work it is particularly desirable to be able to predict any such behaviour for check purposes. Aden (1951) seems to have been the first to suggest that this large-$\alpha$ limit of $\sigma_b$ is just $R$, the ordinary geometrical-optical reflectivity of the scattering material for the case of *normal* incidence on a *plane* surface of the material. Aden gave no proof of this, but his computed values of $\sigma_b$ appeared to be converging towards a limit near $R$ for his case, namely 0·64 for 16·23 cm radiation and water. However, the degree of oscillation persisting at his upper limit $\alpha = 6$ was too large to confirm this point computationally. Herman and Battan (1961, Fig. 2) show $\sigma_b$ for 3·21 cm radiation scattered by water spheres for $\alpha$ as high as 10, providing stronger hint that $\sigma_b$ is steadying down to a large-limit of 0·63, which is $R$ for their case; but even here existence of this limit is only inferential.

The value of $R$ for normal incidence on a plane surface of an absorbing dielectric whose complex index of refraction is $N = m(1 - ki)$ may be calculated from the relation (Stratton 1941),

$$R = [(m-1)^2 + (mk)^2]\,[(m+1)^2 + (mk)^2]^{-1} \quad . \quad . \quad . \quad (2)$$

The absorption term in $N$ is related to $K$, the conventional Napierian coefficient of absorption, according to $mk = K\lambda/4\pi$; that is, $mk$ gives the Beer's law transmission factor for a thickness of the dielectric equal to the wavelength divided by $4\pi$. Although values of $R$ computed from Eq. (2) do seem to be the limits toward which the $\sigma_b$ curves of Aden and of Herman and Battan are converging in damped oscillatory fashion, there is need for a direct proof. The peculiar nature of the $\sigma_b$ definition makes such an asymptotic rule less than obvious; but in Section 2 the suspected rule will be shown to follow from simple consideration of geometrical optics.

## 2. Large-$\alpha$ limit of $\sigma_b$

For fixed $\lambda$, large $\alpha$ means large $a$. In the limit of infinitely large $a$ the sphere appears to the incident plane waves to be a flat surface, so one is first tempted to assert that $P_s = RP_i$. But this, in Eq. (1) would imply $\sigma_b = 4r^2 R/a^2$, showing something amiss since no clear meaning can be attached to the ratio $r^2/a^2$ in the limit of infinite $a$. (Furthermore, an underlying requirement of the Mie theory is that we confine our attention to the far-field where $r \gg a$). The fallacy in the preceding requirement is that we have lost sight of the essential geometric role of the spherical shape of our scatterer (which in the large-$\alpha$ limit is just a reflector, of course).

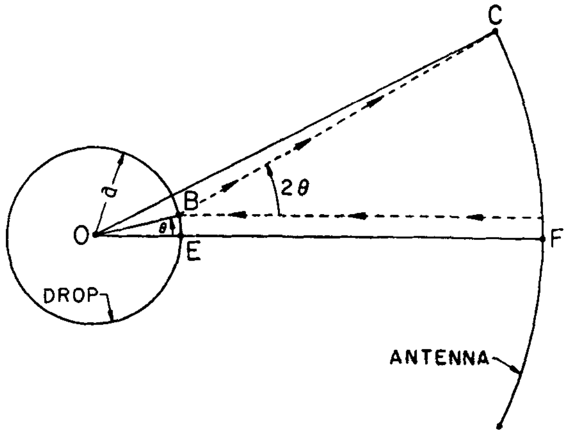


Figure 1. Schematic diagram of back-scattering.

A relevant feature of the radar back-scatter problem is brought out by considering some representative numerical values. In Fig. 1 is shown a schematic diagram of a radar antenna, considered to be a paraboloid of revolution and a scattering sphere for which $\alpha$ is large enough that geometrical optics holds with good accuracy. (The order of $\alpha$ for which the latter assertion is justified depends strongly on the complex index of refraction, as is clear from the work of Herman and Battan). Suppose the circular antenna to have a rim of radius 1 m, and suppose the antenna-to-drop distance FO $= r$ is 10 km, both reasonable values in practice. Then the antenna subtends a half-angle FOC $= 10^{-4}$ radians or a mere 20 seconds of arc, as seen from the drop. But the 'scattered' radiation actually sent from drop to antenna *rim*, following reflection of the incident plane waves, emanates from the point B which, as viewed from O, lies off the axis FO by the tiny angle $\theta$ equal to almost exactly one-half angle FOC, or only about 10 seconds of arc for the cited case. Hence we see clearly that the spherical cap centred on the pole E and having half-angle $\theta$ which contributes every bit of 'back-scattering' reaching the antenna in the large-$\alpha$ limit of specular reflection is so small as to ensure very high accuracy when we treat it as a plane surface. That is, the reflecting cap departs from flatness by too small amount to render Eq. (2) invalid, a point readily corroborated by inspection of the more general expression for $R$ for arbitrary angle of incidence, as derived in works on electromagnetic theory (Stratton 1941).

Thus the specified spherical cap is essentially a flat disc of area $\pi a^2 \theta^2$ which reflects back into a cone of half-angle $2\theta$ radiation of total power $R\pi a^2 \theta^2 P_i$. The disc-area over which this total power is spread by the time the reflected radiation returns to the antenna acting as receiver is, since $r \gg a$, $\pi (2r\theta)^2$ (the reader is warned to note that Fig. 1 distorts the real geometry), whence

$$P_s = R\pi a^2 \theta^2 P_i / 4\pi r^2 \theta^2 \qquad (3)$$

Inserting Eq. (3) into Eq. (1) gives

$$\sigma_b = R \qquad (4)$$

in confirmation of the limit suggested by the computer results for moderately large $\alpha$. It will be clear from this derivation that any notion that Eq. (4) should hold simply because a very large sphere behaves like a plane surface near its pole is incorrect.

### 3. Role of absorption

Based on experience with Mie calculations for a wide variety of complex indices, Herman's conclusion (oral communication) is that it is primarily the imaginary part of $N$, that is the *absorption* term, that dictates the rapidity of convergence of $\sigma_b$ to its geometric optics limit $R$. The larger $mk$, the more rapidly $\sigma_b$ settles down to an asymptotic approach to $R$. He has recently extended calculations of $\sigma_b$ to $\alpha = 100$ for a specially selected hypothetical case of non-absorbing spheres of index $N = m = 2{\cdot}0$, chosen to yield exact front-surface focusing of paraxial rays on to the rear pole of the spheres. Near $\alpha = 100$ he found that $\alpha_b$ is about 100 and is still rising, whereas from the geometrical-optical limiting theory outlined above we should expect for such an index and so large an $\alpha$, that $\sigma_b$ would be near $R$, which for this case equals 1/9. The physical explanation of this failure is related to the recently clarified point that back-surface internal reflection of radar beams is of major importance in governing $\sigma_b$ for poorly absorbing spheres (Atlas, Harper, Ludlam and Macklin 1960; Herman and Battan 1961).

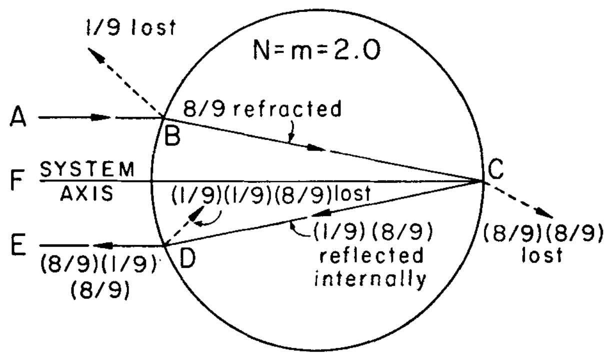


Figure 2. Back-surface internal reflection.

Fig. 2 shows what is happening in Herman's hypothetical case of $N = 2{\cdot}0$. For $\alpha$ near 100 we are essentially in the geometrical optics region, so we may examine, on mere ray-tracing principles, a pencil of paraxial rays AB arriving parallel to the system axis FC. At B the pencil is refracted toward C, suffering a front-surface reflective loss which we may take as about 1/9 because we are actually concerned here only with rays very near FC. The 8/9 of the pencil refracted into the interior of the sphere at B goes on to suffer partial internal reflection at C such that about 1/9 of 8/9 is sent back along CD, the other 8/9 of 8/9 being lost as it refracts out of the sphere's back surface as indicated. At D an internal reflective loss occurs as shown, while an emerging pencil DE heads back to the radar set nearly parallel to FC and with intensity of *about* 8/9 (1/9) (8/9) times the incident intensity of the pencil along AB. Despite this roughly ten-fold intensity reduction produced at the three dielectric discontinuities between A and E, this overall process has clearly led to an extremely large contribution to $\sigma_b$, for the all-important feature of the emerging pencil is that it is heading back toward the radar *almost parallel to the system axis.* For incident rays, a small but finite distance off FC (such as AB itself in the exaggerated schematic version in Fig. 1), spherical aberration will produce slight aparallelism, but the back-scattered (back-reflected) intensity *per unit solid angle* near FC will be very high here. Now, although it is not made explicit in conventional definitions of $\sigma_b$ (e.g., Mason 1957; Battan 1959), and is, indeed, obscured in the working equation usually employed (see Eq. (1)), $\sigma_b$ is actually a measure of back-scattered power *per unit solid angle.* Hence the tendency of back-surface reflection to focus many nearly paraxial rays into an *almost* non-divergent returning pencil powerfully enhances $\sigma_b$ *when internal absorption is low enough to preclude appreciable attenuation on the round-trip between front and back surfaces.* With a fairly low value of $mk$, and even with $m$ being somewhat smaller than the value 2·0 that gives paraxial focus at the rear pole, one may anticipate a general tendency towards very high $\sigma_b$, for then some annular array of incident pencils arriving at a *finite* distance off the system axis will have their foci at or very near C, yielding again the geometry of Fig. 1. This probably explains Herman and Battan's Fig. 2 where ice of $N = 1{\cdot}78\text{-}0{\cdot}0024i$ is found, for $\lambda = 3{\cdot}2$ cm, to have $\sigma_b$ near 30 for $\alpha$ near 30, whereas $R = 0{\cdot}08$ for that $N$. Most real dielectrics will have $k$ non-zero, even if small, so there will be some range of sufficiently large $\alpha$ that $\sigma_b$ is not significantly different from $R$. That range could, of course, be associated with absurdly large sphere radii, but at least the limit exists.

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*NOTE*: Following completion of the above manuscript it was pointed out to the writer by Dr. D. Atlas that the large-$\alpha$ limit of $\sigma_b$ as given by the relation (4) derived here, is familiar to workers concerned with lunar-radar echoes (the moon functioning just as does a large drop, but to obviously greater degree of precision). Inquiry has not disclosed any treatment in the literature giving the above proof, and the latter, as well as the limit (4) itself, are evidently not widely known to radar meteorologists.