# The Saturation Adjustment in Numerical Modelling of Fog[1]

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23 October 1962 and 22 July 1963

In numerical experiments (e.g., Fisher and Caplan, 1963) on the formation of fog under conditions of constant pressure, the principal governing equations will be two continuity statements of form

$$\frac{\partial\psi}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial\psi}{\partial z}\right)-\mathbf{V}\cdot\nabla\psi+S, \tag{1}$$

where $\psi$ is, in turn, some measure of heat content and some measure of water vapor content at time $t$ and height $z$ subject to turbulent diffusion controlled by the field of $K$, the eddy diffusivity for $\psi$, and subject to horizontal advection controlled by the field of the velocity $V$. The last term $S$ is a source term, which constitutes the point of interest of this note. In (1), only vertical diffusion is considered, but greater generality is formally obtainable by obvious changes that are of no concern here.

In each iteration, the appropriate computations at each gridpoint will yield finite-difference equivalents of the first two terms on the right in (1), based on the fields of $\psi$, $K$, and $V$, given by the just-preceding iteration. If the model allows for contributions to $S$ made at the air-earth interface (heating or cooling, evapora-

[1] Work supported by the Office of Naval Research.

tion or condensation), those $S$-effects will be appended to the diffusion and advection effects according to (1), and the resultant vapor pressure and temperature will then have assumed some values $e_0$ and $T_0$, respectively. When the aforementioned steps have been completed, one further step is still required before that iteration is complete: There will, in general, exist some imbalance between the actual vapor pressure $e_0$ and the saturation vapor pressure $e_{s0}$ corresponding to $T_0$, whence the left member of (1) is not fully prescribed for the given iteration until a "saturation adjustment" has finally been carried out as a part of the contribution of the source term $S$. If the air has been supersaturated, the iteration must be concluded by *condensing* some excess vapor; whereas if the air has been left in a subsaturated state, some of the already-present liquid water must be *evaporated*. In either event, this adjustment process occurs adiabatically and isobarically, and is accompanied by temperature changes that must be predicted to close the iteration.[2]

To my knowledge, the only existing numerical analysis of fog-formation is the recent study by Fisher and Caplan (1963), which gives encouraging evidence of the insights to be gained by numerical experiments on fog-forming and fog-dissipating processes. Those authors' method of handling the saturation adjustment for $S$ in (1) is only an approximation method that does not rest directly on the underlying thermodynamics. A thermodynamically valid basis for adjusting $S$ for saturation imbalance will be derived here and compared with the method of Fisher and Caplan.

One might use either vapor pressure or mixing ratio as the measure of water vapor content. Here, I shall use the former in the interest of simpler physical interpretation, but it will be obvious how the analysis and final saturation adjustment relation is to be altered if the latter humidity measured is preferred. The derivation will be described in terms applying to the particular case where $e_0 > e_{s0}$, i.e., the case in which the diffusion, advection, and surface contributions to the given iteration at the given grid-point have produced an actual vapor pressure which *exceeds* the saturation vapor pressure corresponding to the temperature produced by those same three processes at the given grid-point. The final relations are the same, however, whether we are adjusting for supersaturation or subsaturation, so long as due regard is paid to signs involved.

In Fig. 1, the point $P_0$ with coordinates $e_0$, $T_0$ represents the state of our parcel *after* making allowance for diffusion, etc., but before saturation adjustment, and lies above the saturation vapor pressure curve, as shown. Our adjustment must involve condensing onto the already present fog drops the proper amount of this

[2] There is some similarity between the present saturation-adjustment in fog calculations and a step that must be incorporated into numerical weather prediction models involving condensation processes (e.g., Smagorinsky, 1956).

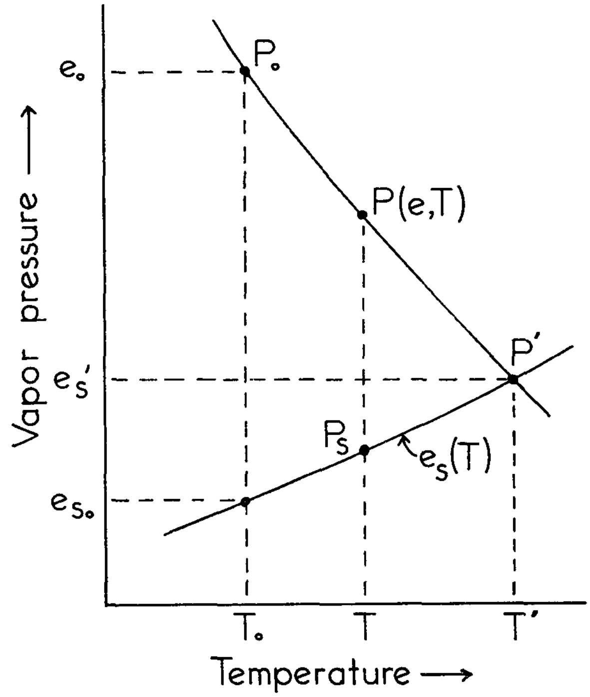


FIG. 1. Graphical interpretation of the process of saturation-adjustment.

excess vapor, where "proper amount" carries the following important connotation: The condensation will itself raise $T$ by virtue of released latent heat, so we must find a way of describing the *mutual approach* of our characteristic point $P(e,T)$ and $P_s(T)$, the latter being the variable point on the saturation curve at temperature $T$ as the latter rises during adjustment.[3]

If we call the mixing ratio $w$ and the latent heat of vaporization (at temperature $T_0$) $L$, and if the specific heat at constant pressure is $c_p$, then the heating $dT$ accompanying a small amount of adiabatic-isobaric condensation is given by $-Ldw_s = c_p dT$; and since $w_s \doteq \epsilon e_s/p$ where $\epsilon = 0.622$ and $p$ is the pressure, this is equivalent to

$$-(L\epsilon/p)de_s = c_p dT, \qquad (2)$$

which specifies the locus along which the characteristic point $P$ moves during the condensational heating. In Fig. 1, this locus is indicated as the nearly straight line running through $P_0$ towards the lower right.

As $P$ moves down this locus, $T$ is of course rising, and hence $P_s$ moves up along the saturation vapor pressure curve. Where $P$ and $P_s$ meet at $P'$ is the desired

[3] The reader should note the interesting analogy between this problem and the psychrometric problem in the diffusional growth of cloud particles (Marshall and Langleben, 1954). The temperature $T'$ defined here may be viewed as a generalized wet-bulb temperature (generalized to embrace supersaturated as well as subsaturated air); and equation (4) can be regarded as a generalized psychrometric relation.

final (just-saturated) state of our parcel, whose co-ordinates there we shall call $e_s'$ and $T'$.

By the Clausius-Clapeyron equation for the saturation vapor pressure curve, we have the very good approximation (since $T$ is changed only slightly in any given iteration's saturation-adjustment)

$$(e_s'-e_{s0})/(T'-T_0)=(\epsilon L e_{s0})/RT_0^2, \tag{3}$$

where $R$ is the gas constant for dry air. On the other hand, (2) implies, to a similarly good approximation, that

$$e_0-e_s'=(c_p p/\epsilon L)(T'-T_0). \tag{4}$$

Inspection of Fig. 1 reveals that

$$e_0-e_s'=(e_0-e_{s0})-(e_s'-e_{s0}). \tag{5}$$

Inserting (5) into (4) and making use of (3) yields a relation which, when solved for $(T'-T_0)$, gives us as the principal working equation for accomplishing the saturation adjustment

$$T'-T_0=A(e_0-e_{s0}) \tag{6}$$

in which $A$ is a function of $T_0$, given by

$$A=(\epsilon LRT_0^2)/(c_p pRT_0^2+\epsilon^2L^2e_{s0}) \tag{7}$$

and $L$ is to be evaluated, in each iteration, at $T_0$.

After the preliminary diffusion, advection, and surface-source effects have been allowed for at the given grid-point, the resulting values of $e_0$ and $T_0$ suffice to determine $T'$ through (6) and to determine the final (saturated) vapor pressure $e_s'$ by (3), while the amount of condensate to be partitioned among the fog drops in the parcel is governed by (4). *Thus* (3), (4), *and* (6) *are the desired equations governing the saturation adjustment needed to complete each iteration in a fog-model calculation.*

By contrast, Fisher and Caplan's approach corresponds to replacing (6) by

$$(T'-T_0)=B(e_0-e_{s0}) \tag{8}$$

with

$$B=(RT_0^2)/(\epsilon L e_{s0}). \tag{9}$$

Comparative values of coefficients $A$ and $B$, shown for three representative temperatures in Table 1, indicate

TABLE 1. Values of $A$ and $B$ (units of deg C per mb) for three temperatues at $p=1000$ mb.

| $T$ (deg C) | $A$ | $B$ |
|---|---|---|
| 0 | 0.92 | 2.26 |
| 20 | 0.47 | 0.69 |
| 40 | 0.22 | 0.25 |

that Fisher and Caplan's saturation adjustment scheme would overestimate the temperature-change by substantial amounts, especially at low temperatures, although at their working temperature of 15C, it gave fairly adequate results. Since the thermodynamically correct method here outlined can be quite easily programmed into machine computations, it is urged that it be used in future numerical studies on fog.

## REFERENCES

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