Effect of Momentum-Exchange on the Fall Velocity of an Accreting Hydrometeor[1]

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When a raindrop (or any other hydrometeor) is growing by accretion as it falls through a cloud of smaller drops, each collision and coalescence involves a momentum-exchange process. Under typical conditions, the velocity and mass of the *raindrop* are each several orders of magnitude greater than those of the accreted *cloud drops*, so it is essentially correct to view the process as one in which small particles initially at rest with respect to the air are collected and impulsively given the relatively large velocity of the raindrop. Although the gain in accreted mass tends to produce an increased velocity, the accompanying momentum-exchanges tend, conversely, to *retard* somewhat the velocity of the raindrop itself, since the momentum of the latter is being repeatedly diluted by addition of small masses of zero momentum. That is, accretion involves a dynamical process inverse to that which produces the thrust of a rocket. Since all laboratory measurements of terminal velocities are made in cloud-free air the following question arises: *Does this momentum-sharing process reduce velocities of accreting hydrometeors by an amount large enough to require adjustment of the conventionally used laboratory terminal velocities?* This rather interesting dynamical question, which does not appear to have received previous consideration, will be answered below.

The equation of motion for a raindrop of mass m falling with speed v (taken positive downward) and accreting, at rate dm/dt, cloud drops of essentially zero initial relative momentum is

$$m\frac{dv}{dt}+v\frac{dm}{dt}=mg-D, \tag{1}$$

where t is time, g is the acceleration of gravity, and D is the aerodynamic drag force. If the raindrop has radius r and drag coefficient C_d and is falling in air of density ρ_a, then $D=(1/2)\rho_a v^2 C_d \pi r^2$. If s is the distance of fall, measured positive downward as for v, then from purely geometric considerations we have $dr=(W/4\rho_w)ds$, where W is the *collectable* cloud liquid water content for the raindrop, and ρ_w is the density of

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water. Using these relations in (1), the equation of motion may be put into the form

$$\frac{dv}{dt}+[(3\rho_a C_d)/(8\rho_w r)+(3W)/(4\rho_w r)]v^2=g. \qquad (2)$$

Physical considerations suggest, *a priori*, that an accreting drop will, as in the case of a drop in cloud-free air, move at all times under conditions of almost balanced forces such that the acceleration term in (2) can be put equal to zero. We make this assumption, for it will be shown below to be acceptable. If we designate as V_t the terminal velocity in *cloud-free air* for a raindrop of the same size as that here considered, we may simplify (2) by use of $V_t^2=8\rho_w gr/3\rho_a C_d$, thereby obtaining for the terminal velocity v_t of our accreting raindrop the final relation

$$v_t=V_t(1+2W/\rho_a C_d)^{-\frac{1}{2}}. \qquad (3)$$

The preceding derivation of the terminal velocity v_t in the presence of momentum-exchange effects has merit of both mathematical simplicity and ease of physical interpretation. Whether appreciable error might be introduced by neglect of the acceleration effects can only be determined from a complete solution to (2). Although (2) can be put into a form linear in the quantity v^2 by changing from t to s as independent variable, one thereby only succeeds in reducing the problem to quadrature since a required integral cannot be found. However, putting $v=ds/dt$, expanding s in a power series about $t=0$ with the intial conditions $r=0$, $v=0$ at $t=0$ (whence one has $r=Ws/4\rho_w$), yields the closed solution

$$s=gt^2/2(7+3\rho_a C_d/W) \qquad (4)$$

from which we find $v=gt/(7+3\rho_a C_d/W)$ for the instantaneous velocity as a function of time of fall. Hence, introducing the laboratory terminal velocity V_t in the manner previously employed gives

$$v=V_t(1+7W/3\rho_a C_d)^{-\frac{1}{2}} \qquad (5)$$

as an *exact* velocity solution, for the stipulated initial conditions, with acceleration taken into account.

Comparison of solutions (3) and (5) reveals that the exact solution (5) predicts a slightly *smaller* fall-speed than does (3), the numerical coefficient of the second term (correction term) within the parenthesis changing from 2 to 7/3. This roughly 17 per cent change in the correction term is, however, a change in a term small compared with unity, as will be noted below; so the two solutions prove to be numerically almost identical for meteorologically likely values of all parameters involved. The fact that allowance for acceleration effects gives a *decreased* speed is quite understandable, since the speed of an accreting drop must always tend to lag slightly behind the velocity value implying instantaneous balance of the weight, drag, and momentum-exchange forces assumed in deriving the approximation (3).

Two assumptions employed above require brief comment:

(a) Assuming *continuity* of accretion in writing the term $v(dm/dt)$ in (1) might be questioned in view of the fact that mass is added in *discrete* amounts in the real case. That this approximation *is* acceptable, however, follows jointly from the typically very small ratio of masses of cloud droplet and raindrop, and from the circumstance that the average fall-distance between collisions (order of 10^{-1} cm in cumulus) is very small compared with the relaxation distance (order of a few *meters*) for velocity-adjutsment to a step-function change in raindrop mass, as can be shown by a somewhat lengthy analysis that will not be reproduced here.

(b) The use of the *meteorologically* rather unrealistic initial conditions of zero mass and velocity at $t=0$ in obtaining an exact solution in the simple form of (4) is here acceptable because the ultimate motion proves to be quite insensitive to the initial conditions, as is in fact, shown by the near-equality of (3) and (5).

By twice differentiating both sides of (4) with respect to time we obtain

$$d^2s/dt^2=g/(7+3\rho_a C_d/W). \qquad (6)$$

Thus we find, somewhat surprisingly, that the acceleration is *independent of time* so long as ρ_a, W, and C_d may be regarded as sensibly constant, which is often tolerably well satisfied in real clouds. If the quantity ρ_a/W approaches *zero*, the acceleration as given by (6) approaches asymptotically the value $g/7$, a result implied by Becker (1954, p. 189), which result was what first led the writers to examine the possible meteorological importance of momentum-exchange effects in accretion processes. Note that ρ_a/W can become small either through diminution of ρ_a (whence aerodynamic drag simply vanishes) or through increase of W (whence momentum-exchange forces overwhelm drag effects in controlling the dynamics of the accreting drop). We point out these limiting properties only in passing, for they will scarcely be important in real clouds.

As had been inferred above from physical considerations, (3) or (5) imply that momentum-exchange *lowers* the velocity of accreting drops below the laboratory values of the terminal velocity for drops of the same radius r falling in cloud-free air. A question of meteorological interest is whether this reduction is small or large compared with the experimental accuracy with which V_t can be measured. For a typical raindrop with radius of about 1 mm (and hence with drag coefficient very near 0.5) falling at the 600-mb level through a cloud of collectable liquid water content 2 g/m^3, (3) or (5) implies a reduction of terminal velocity of *almost exactly 0.5 per cent.* By way of comparison, the best laboratory measurements of V_t, those of Gunn and Kinzer (1949) involve an experimental accuracy stated to be 0.7 per cent over the entire drop-size range which those workers considered. Inasmuch as the accuracy attained by those investigators was poorest for small drops and best for drops of raindrop size, we may suppose that their *raindrop* V_t values are accurate to at least 0.5 per cent. Thus momentum-exchange effects on accreting raindrops lower v_t or v below V_t by an amount that happens to be *about equal to the experimental uncertainty in* V_t *itself.*

From the above we see that past neglect of momentum-exchange in accretional-growth computations represents an error that is not serious compared with other uncertainties involved. At the same time, however, it appears that momentum-exchange is a dynamical factor of at least conceptual interest since its effects are not trivially minute as measured against the yardstick of available V_t-accuracy. Inspection of (3) or (5) reveals that the retarding effect of accretional momentum-exchange will, however, rapidly become quite insignificant as one goes from raindrops down to drizzle drops or to large accreting cloud drops, for the *collectable* liquid water content rather rapidly diminishes, and at the same time the effective drag coefficient quickly rises, as r falls below ordinary raindrop size. For *small* hailstones that are accreting supercooled drops, the effect will be of about the same magnitude as for raindrops, since C_d remains close to 0.5 in the Reynolds number range corresponding to moderate hail sizes. However, for *very large* hail, transition to a turbulent boundary layer can reduce C_d to as little as 0.2, roughly doubling the significance of momentum-exchange effects. Thus for large hail accreting supercooled drops in a cumulus with very high liquid water content, the process described here will lower terminal velocities by *a few per cent.*

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