watch the technologies sprung from our science, we will not remain secure and worthy among men.

The day will come, I am hopeful, when such a meeting as this one is viewed by the world as mainly a gathering of teachers, of man and women who construct great new ideas and display older ones in the spirit of Einstein, and not mainly of those who smash the nucleus to explode new bombs. Until that day arrives, the legacy of Oersted and of all those colleagues across the millennia on whose work we stand will not have been wisely or humanly employed.

Maxwellian Interpretation of the Laplacian

J. E. McDonald
Institute of Atmospheric Physics, The University of Arizona, Tucson, Arizona
(Received 22 March 1965)

In many physical relationships involving the Laplacian of a function, it proves useful to interpret the Laplacian as a measure of the *local anomaly* of that function. This little used, but often very illuminating interpretation, due originally to Maxwell, is illustrated with a number of examples drawn from mathematical physics.

I. INTRODUCTION

The purpose of the following discussion is to describe certain features of a pedagogically very useful bit of nearly lost knowledge. It concerns a simply derived and easily understood interpretation of the Laplacian function $\nabla^2\varphi$ of any scalar φ. With the aid of this interpretation, it is possible to describe more clearly the physical meaning of the many equations of mathematical physics which involve the Laplacian.

The interpretation seems to be nearly lost knowledge in the sense that it goes unmentioned in almost all texts on mathematical physics. The basic point developed below, however, is found in a little volume by Hopf,[1] and a short discussion of the principle is given by Morse and Feshbach.[2] Hopf's examination of the matter has also been cited by Sutton.[3] Aside from those few references, I know of no other mention of the point in the current literature, despite its evidently strong heuristic and pedagogic value.

It had been my impression that the principle involved was due to Hopf until I recently came upon a brief remark on the matter in the classical hydrodynamics treatise of Lamb.[4] As Lamb notes, the interpretation is originally due to Maxwell.[5] Hence this note may be said to concern the Maxwellian interpretation of the Laplacian, an interpretation whose usefulness I try to explain and illustrate in somewhat more detail than that to be found in any of the cited references.

II. THE LAPLACIAN AS A MEASURE OF LOCAL ANOMALY

Let φ be any scalar continuous function of position with value φ at point 0. We wish to determine the average value of φ in the immediate neighborhood of 0. Where Maxwell computed that average over a sphere concentric with 0, Hopf computes it over a small cube, of edge-length h, centered on 0. The latter method yields the same important property pointed out by Maxwell and is somewhat more easily followed than the spherical-coordinate analysis evidently used by Maxwell.

[1] L. Hopf, *Introduction to the Differential Equations of Physics* (Dover Publications, Inc., New York, 1948), pp. 62–68.
[2] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Publishing Company, Inc., New York, 1953), pp. 6–8. This analog of Eq. (2) of the present paper contains an incorrect exponent, it should be noted.
[3] O. G. Sutton, *Mathematics in Action* (Harper & Brothers, New York, 1960), p. 66.
[4] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), p. 577.
[5] J. C. Maxwell, *Scientific Papers* (Dover Publications, Inc., New York, 1952), p. 264. This is the paper in which Maxwell coined the term "curl," plus several other terms that have not survived, such as "slope" for what we now term "gradient" and "convergence" for the negative of what we now term "divergence."

We place the origin of a rectangular xyz-coordinate system at 0 with axes parallel and perpendicular to the cube edges, so that the various cube faces lie at distances $-\frac{1}{2}h$ and $\frac{1}{2}h$ from the origin. To obtain the average of φ over the cube, we may, assuming all required continuity of φ, expand φ in a Taylor series about the origin and integrate the series over the volume of the cube to get

$$\bar{\varphi}=\frac{1}{h^3}\iiint_{-h/2}^{h/2}\varphi(x,y,z)dxdydz=\frac{1}{h^3}\iiint_{-h/2}^{h/2}\Big[\varphi_0+\left(\frac{\partial\varphi}{\partial x}\right)_0 x+\left(\frac{\partial\varphi}{\partial y}\right)_0 y+\left(\frac{\partial\varphi}{\partial z}\right)_0 z$$
$$+\tfrac{1}{2}\left\{\left(\frac{\partial^2\varphi}{\partial x^2}\right)_0 x^2+\left(\frac{\partial^2\varphi}{\partial y^2}\right)_0 y^2+\left(\frac{\partial^2\varphi}{\partial z^2}\right)_0 z^2\right\}$$
$$+\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)_0 xy+\left(\frac{\partial^2\varphi}{\partial x\partial z}\right)_0 xz+\left(\frac{\partial^2\varphi}{\partial y\partial z}\right)_0 yz+\cdots\Big]dxdydz. \quad (1)$$

Because of the symmetric limits used in each integration, odd functions do not contribute to the integral; hence in the limit of h small enough to neglect contributions of fourth, sixth, and other higher even derivatives in the Taylor series, one has

$$\bar{\varphi}=\varphi_0+\frac{h^2}{24}(\nabla^2\varphi)_0,$$

whence

$$(\nabla^2\varphi)_0=\frac{24}{h^2}(\bar{\varphi}-\varphi_0). \quad (2)$$

Note that the presence of h^2 in the right member merely reflects the fact that the amount by which the cube-average $\bar{\varphi}$ departs from the value φ_0 at the cube center is dependent upon the size of the cube. As h shrinks towards zero, $\bar{\varphi}$ approaches φ_0, of course. Maxwell[5] obtained an expression equivalent to (2), but with $10/r^2$ as the prefactor on the right side, where r is the radius of a small sphere inscribed about point 0. (A good problem for students: Confirm Maxwell's prefactor $10/r^2$.)

Equation (2) permits one to say that *the Laplacian of a function at point 0 is a measure of the local anomaly of that function relative to the function's average in the immediate neighborhood of 0.*

It proves convenient to borrow a term which geophysicists use in describing irregularities in gravitational potential and thereby to formalize the preceding statement by expressly defining the *local anomaly* of φ at 0 to be the limit of the right member of (2) as h approaches zero. So defined, the local anomaly may be positive, or zero, or negative. The latter case arises when φ_0 exceeds the average $\bar{\varphi}$ in its neighborhood. Having made this definition, we can say that the Laplacian of φ at 0 *equals* the local anomaly at 0, in the sense of (2).

An alternative conception that would pay more direct attention to signs involved would result from defining the right member of (2) as the "local deficit" of φ at 0; but one finds that the required circumlocution in speaking of excesses of φ at 0 as "negative deficits" tends to be somewhat confusing whence the above-mentioned definition of the local anomaly seems preferable. Interestingly, Maxwell, in his 1871 paper, was able to avoid this minor problem of signs because he defined the "operator of Laplace" to be the negative of our modern Laplacian operator. Whether this was an *ad hoc* definition on the part of Maxwell or a then current convention, I do not know; but it had the virtue of permitting Maxwell to give $\nabla^2\varphi$ the name, *concentration*, a rather felicitous term, for it carries its own sign implications (as does also the awkward term "local deficit," whereas "local anomaly" is intentionally noncommittal as to signature).

III. EXAMPLES OF THE LOCAL ANOMALY INTERPRETATION

Several of the following examples were suggested by Hopf[1] and some are briefly mentioned by Morse and Feshbach.[2] Those, plus additional examples, are examined here in terms of the concept of the local anomaly. Maxwell, in his short 1871 paper, which dealt mostly with nomenclatural matters, gave no physical illustrations of the interpretation which he placed on the Laplacian.

1. Poisson's equation. If φ is the electrostatic potential at a point where the charge density has value ρ, then Poisson's equation may be written, with suitable choice of units, as

$$\nabla^2\varphi = -4\pi\rho. \tag{3}$$

In view of the conventions and definitions given in the preceding section, we may now say that (3) asserts that the local anomaly of the electrostatic potential is given by -4π times the local charge density, which has the meaning that presence of charge at point 0 is responsible for distorting (producing anomalies in) the potential in its immediate neighborhood. Positive charges, according to (3) create negative local φ-anomalies, whence (2) implies that φ at a point 0 occupied by a positive charge is necessarily larger than $\bar{\varphi}$ in the h neighborhood of 0. That the latter assertion holds independently of the sign and density of all other charges in the h neighborhood is a point that may require some clarification for students who have not thought of Poisson's equations in these terms before. A number of simple cases prove effective therein: isolated point charge, uniform line charge, a roomful of charge of one sign and uniform density, etc. Certain seemingly simple cases quickly bring out requirements of continuity and of boundedness that place certain restrictions on (2). Summarizing, we say that Poisson's equation, under the interpretation being discussed here, implies the eminently sensible rule that charges produce local anomalies in the spatial distribution of the electrostatic potential. Poisson's equation is a useful first example to cite because the anomaly interpretation is for it extremely obvious, indeed almost trivially obvious.

If in (3) we put $\rho=0$ we have Laplace's equation, which holds in many physical systems. On the present viewpoint, we say that Laplace's equation asserts that for source-free fields the local anomaly vanishes everywhere, a condition implying great "smoothness" of the φ-field. The latter property of harmonic functions will be briefly examined later.

2. Equation of heat conduction (and diffusion). If φ is the temperature in a medium of uniform thermal diffusivity k, free from sources, then the heat equation is

$$\partial\varphi/\partial t = k\nabla^2\varphi. \tag{4}$$

Here we read (4) as saying that the local time rate of change of temperature at 0 is controlled by the magnitude and sign of the local anomaly of temperature in the h neighborhood of 0. If the local anomaly is positive, the temperature will be rising at 0. Equation (2) relates this to elementary physical considerations, since a positive local anomaly of temperature implies that the element of the medium at 0 is, on the average, surrounded by hotter portions of the medium, and the latter must, on the whole, be sending heat towards 0.

Alternatively, if φ be regarded as the concentration of some diffusing substance, say water vapor in the air, and if k be the constant vapor diffusivity for this gas mixture, then (4) becomes the diffusion equation, and we interpret it as showing how the vapor concentration is rising or falling at a point depending on the value of the local anomaly of the vapor concentration at that point. (We are at an advantage over Maxwell here; he would have to cope with the notion of the concentration of the concentration!) As before, this interpretation carries immediate and sensible physical meaning if one is accustomed to think in terms of (2). For if, say, the local vapor anomaly is *negative* at 0 (i.e., if according to (2), the vapor concentration at 0 is *higher* than the average over the entire h neighborhood), then elementary diffusion considerations imply that vapor will be, in the net, flowing out of the volume element at 0, whence the local concentration at 0 must be *decreasing* with time, as (4) predicts.

Although there are other physical systems obeying (4) which might also serve as illustrations, the above two should suffice to show that the local anomaly interpretation of the Laplacian is particularly meaningful when applied to any of the broad class of diffusion equations of form (4).

3. Wave equation. If φ is, speaking now somewhat loosely and broadly, any measure of local strain or distortion in a medium possessing both elasticity and inertia, then distortion waves can propagate with speed c through that medium according to the relation

$$\partial^2\varphi/\partial t^2 = c^2\nabla^2\varphi, \tag{5}$$

assuming uniform mechanical properties of the medium. Although the three-dimensional case of (5) is not different in essence from the two- or the one-dimensional case, we obviously have a more easily interpreted situation if we examine, say, the two dimensional case of waves on an elastic membrane of nonzero surface density. Then (5) describes how the "acceleration" of the distortion coordinate φ is directly proportional at each point to the local anomaly $\nabla^2\varphi$ instantaneously existing there. Thus, suppose point 0 in the membrane is, at a specific instant, *above* the equilibrium plane so that φ is there positive, and suppose furthermore that the adjacent membrane topography implies a negative local anomaly such as would exist, for example, at the summit of a temporary bulge in the undulating membrane. Then by (5), one infers that this negative anomaly must be tending to diminish (algebraically) the local membrane velocity $\partial\varphi/\partial t$. This deceleration may mean that the membrane is moving back towards the equilibrium configuration with increasing speed, that its outbound speed is being decreased, or perhaps that its velocity is just being reversed at a positive extremum. Clearly, the local anomaly view of the Laplacian renders wave equations of form (5) immediately reasonable.

Hopf[1] calls attention to a very important point concerning a basic difference between wave equations of form (5) and diffusion equations of form (4). Although each describes the way in which local anomalies tend to smooth themselves out with passage of time, (4) has the smoothing action applied to what may be loosely termed a "velocity," whereas (5) has the local anomaly controlling the system's "acceleration." Inherent in this difference is the fact that wave equations wave, while diffusion-type equations only yield a monotonic relaxation back towards equilibrium, without overshoot of the type required to set up undulations. The difference is traceable, of course, to the action in (5) of inertial effects that have no counterparts in (4). But the two equations do have one exceedingly important common property, namely each depicts how certain types of local anomalies have a certain tendency to eliminate themselves. That certain extensions of the latter viewpoint have interesting applications in other physical systems describable by Laplacians are obvious to the reader.

4. Hydrodynamical interpretations. A number of useful applications of the local anomaly interpretation of Laplacians can be found in hydrodynamics. The wave equation does, of course, arise there, and Laplace's equation holds for the velocity potential in irrotational flows in two dimensions, so local anomaly interpretations will be found directly useful in both of those cases. Others are now mentioned.

A purely kinematical relationship often used in analysis of two-dimensional rotational incompressible flows relates the stream function ψ to the local vorticity η according to

$$\eta = \nabla^2\psi. \tag{6}$$

This seems to tell us that we only find vorticity in regions of the flow that are characterized by nonzero local anomalies in the field of the stream function. With due attention to the way in which the velocity components u and v in the x and y directions are related to the functions ψ and φ, namely according to

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \quad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

one sees that the local anomaly interpretation brings out the point that we can only expect to find vorticity (in the sense of cross-stream velocity shear and/or curvature of streamlines) where the ψ-distribution is locally characterized by bulges or hollows of the type implying either positive or negative local anomalies of ψ. Simple geometric illustrations of this point are easily contrived.

Frequently, of course, a term involving a Laplacian is only one of several in a hydrodynamic equation. In such cases it seems often to be the one term whose physical import is not clearly grasped by students. Consider, for instance, the Navier–Stokes equation for incompressible, viscous fluids, as used in, say, the analysis of Stokes's problem of the slow motion of a sphere. The acceleration du/dt in the x direction is given by the sum X of the external forces, the pressure-gradient force per unit mass in the x direction, $-(1/\rho)(\partial p/\partial x)$, and a term involving the kinematic viscosity ν and the Laplacian of the u component, i.e.,

$$du/dt = X - (1/\rho)(\partial p/\partial x) + \nu\nabla^2 u. \qquad (7)$$

Of the terms on the right, the last is often regarded, by students, as having somewhat obscure physical meaning, yet the local anomaly interpretation of Laplacians makes that last term quite understandable. Because action of molecular viscosity tends to smooth out velocity gradients and shears, it follows that local velocity anomalies will be smoothed out at rates dependent on the very intensity of those anomalies. [Indeed, it is important to recognize that the viscous term in the Navier–Stokes equation is effectively just a diffusion term fully equivalent in nature to that on the right side of (4) above.] Next, it must be emphasized that the *sense* in which viscous drag forces act on a given fluid-element is also made clear by invoking the local anomaly interpretation: Suppose the local u anomaly is positive near point 0, in the sense of (2). Then the element centered on 0, being surrounded, on the average, by fluid moving faster than it is in the x direction, receives net positive x momentum by viscous exchanges with all the neighboring elements of fluid, and this generates a positive drag force. Similarly, a negative local anomaly of any velocity component must give rise through viscosity effects to a negative drag effect on that particular component.

In other hydrodynamical contexts, and especially in meteorological applications of hydrodynamics, one encounters still other Laplacian terms that are readily grasped when one views them on the local anomaly basis. Very often these are of the general nature of diffusion terms, with eddy diffusivity K replacing the molecular kinematic viscosity appearing in Eq. (7) above. Thus, in the theory of convection[6] one derives a relation showing the way the local time rate of change of vorticity depends on vorticity advection, on horizontal gradients of buoyancy forces, and on turbulent diffusion of vorticity. The latter effect is described by a term $K\nabla^2\eta$, whose meaning seems, at least to the writer, to be rather obscure on anything but the local anomaly interpretation. The latter view, however, permits one to see clearly that η for a given fluid element is locally increased by turbulent diffusion, since vorticity is flowing into the given volume element if, and only if, it lies in a region of positive local vorticity anomaly.

A Laplacian term whose significance is essentially the same as that discussed in the preceding paragraph is also encountered in what Schlichting[7] calls the "vorticity transport" form of the Navier–Stokes equations for incompressible viscous fluids, namely

$$D\eta/Dt = \nu\nabla^2\eta,$$

where ν is the kinematic viscosity of the fluid and the operator D/Dt is the Stokes substantial derivative following the fluid parcel. Here the same interpretation of diffusive exchange of vorticity in regions characterized by local vorticity anomalies applies, but now with the diffusion controlled not by eddy diffusion but by molecular diffusion.

SUMMARY

The local anomaly interpretation of Laplacians, due originally to Maxwell, has been shown above to have value as a device for clarifying the meaning of the numerous relations in mathematical physics that involve a variety of types of Laplacian functions. In view of the usefulness of this interpretation, it seems surprising that it goes almost unmentioned in current texts.

The reader interested in this matter is particularly urged to examine Morse and Feshbach's treatment. They begin with a useful analogy between three-dimensional Laplacians and one dimensional second derivatives, and though their elaboration of what is essentially the same as the present local anomaly viewpoint is brief, it contains some very effective descriptions (e.g., they speak of the Laplacians as measures of "bulginess" in two dimensions and of "lumpiness"

[6] Y. Ogura, J. Atmos. Sci. **20**, 407 (1963).

[7] H. Schlichting, *Boundary Layer Theory* (Pergamon Press, Inc., New York, 1955), p. 54.

in three). Also, Hopf's compact discussion of Laplacians and Sutton's very brief comments warrant attention.

It should perhaps be mentioned here that there is an interpretation of Laplace's equation that is utilized fairly frequently in the literature and that bears interesting relation to the much more general interpretation of Laplacian functions examined above. It can be shown, e.g., by Margenau and Murphy,[8] that any harmonic function φ which satisfies Laplace's equation $\nabla^2\varphi=0$, under given boundary conditions on a specified region of space, is the function which has the least mean-square-gradient of all possible functions meeting the boundary conditions. (Briefly, Laplace's equation is the Euler equation for this particular minimal problem in the calculus of variations.) Thus functions satisfying Laplace's equation are often identified as the *smoothest possible functions* for the given boundary conditions, a description having obvious relations to the present description of those same functions as having everywhere zero local anomaly. Furthermore, such functions, with their high degree of smoothness, are by another important theorem, functions which cannot possess extrema anywhere within their region of definition. Here, again, we have an important relationship that can be given illuminating interpretation through the concept of the local anomaly.[9]

There has been a century of relative neglect of a distinctly useful idea briefly mentioned by Maxwell in 1871. The point of these remarks has been to urge that the Maxwellian interpretation of the Laplacian deserves active revival in the physics classroom and in the physics textbook.

[8] H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., New York, 1943), p. 203.

[9] Note carefully that the preceding two properties are limited to functions satisfying the special requirement $\nabla^2\varphi=0$. Many other important functions whose Laplacians are not identically zero do *not* exhibit the minimal mean-square-gradient property nor the cited extremum property, yet are usefully interpreted on the local anomaly viewpoint.