a case where four particles are involved. As in the two-particle case, the energy, momentum, and mass of each individual particle is represented as a right triangle, i.e., $\mathbf{D}_i\mathbf{D}_{i-1}=\mathbf{P}_i$, $C_iD_i-1=E_i$, $C_iD_i=m_i$, and also

$$\mathbf{D}_0\mathbf{D}_4=\mathbf{P},\ \sum_{i=1}^{4} C_iD_{i-1}=E;\quad i=1,2,3,4.$$

By allowing C_1, C_2, C_3, C_4, D_1, D_2, and D_3 to vary their positions in space while maintaining the above set of relations all physically allowed distributions of energies and momenta can be obtained.

(3) Determine the threshold energy required for producing particles (of mass $m_1, m_2 \ldots m_n$) in the final state by colliding a beam of particles of mass m_b with a target particle of mass m_t. The geometrical solution is shown in Fig. 4 where

$$OM=\sum_{i=1}^{n} m_i,\ OB=m_b.$$

The point A can be easily located on the X axis by trial and error such that $AM=AB+m_t$. Then AB is the threshold energy (K.E. $=AB-M_b$) and OA the corresponding momentum. The requirement of minimum energy insures that all energy-momentum triangles for the individual final particles are similar.

Extension of this method to include composite particles such as nuclei or hyperfragments is straightforward.

It is apparent from the figures that mechanical models (using strings and rods, for example) can be easily constructed for solving these types of problems.

Caustic of the Primary Rainbow

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An instructive problem in the geometrical optics of caustics is afforded by the classical theory of the rainbow. Since any caustic may also be viewed as a cusp locus of wavefronts, the same problem has bearing on certain aspects of the diffraction theory of the rainbow. Using a method of caustic analysis capable of delineating the complete emergent caustic of the primary rainbow, it is shown that past conceptions of the full caustic locus have been markedly in error. The correct locus is found to exhibit an odd cusp on a segment of the (virtual) caustic lying wholly within the raindrop.

THE following note corrects a long-standing error, due originally to Potter[1], in the conception of the caustic (cusp locus) of the primary rainbow. Although the past error involves portions of the emergent caustic enveloping rays far enough from the Cartesian ray that the error does not sensibly influence the Airy approximation, the corrected caustic geometry deduced below is so different from the previous version as to warrant brief discussion. Also, the methods employed below have further application in elucidating certain features of the caustic that are quite relevant to the diffraction theory of the bow, yet are not to be found in such standard references as Pernter–Exner[2] and Humphreys.[3]

In the definition sketch of Fig. 1, wavefront ACT has just become fully emergent with a cusp at C, whence C is a point on the caustic (indicated as a dashed curve, but not carried all of the way to the drop surface since the manner of contact there is the point of past error). Where the Cartesian ray strikes the drop at angle of incidence I is shown another ray, a small distance above it, incident at angle i. The latter ray

[1] R. Potter, Trans. Cambridge Phil. Soc. 6, 141 (1838).

[2] J. M. Pernter and F. M. Exner, *Meteorologische Optik* (W. Braumueller, Vienna, 1922), 2nd ed.

[3] W. J. Humphreys, *Physics of the Air* (McGraw–Hill Book Company, Inc., New York, 1940), 3rd ed.

undergoes over-all deviation D on emerging at angular distance θ below the axial line. This ray must, as must all rays incident at $i>I$, contact the cusp locus tangentially at some point C'. To obtain equations specifying the coordinates of the general point C' as a function of i, we follow Potter. Let O be the center of the raindrop of radius a and let ρ be the radius vector to C' intersecting the arbitrary ray at angle γ. If OQ be taken normal to the dashed backward extension of the emergent arbitrary ray, then $\rho \sin\gamma = a \sin i$. Now if i be varied slightly, then γ varies to same order, but ρ is variable to only higher order inasmuch as any envelope may be regarded as the locus of points in which neighboring members of the enveloped family intersect. Hence, $d\gamma/di = \tan\gamma/\tan i$.

But it is established in the elementary rainbow theory that, if r be the angle of refraction, $D=2(i-r)+n(\pi-2r)$, n being the number of internal reflections, here one. If the index of refraction is μ, Snell's law implies $dD/di = 2[1-(n+1)(\cos i/\mu \cos r)]$. The stationary condition already invoked carries the corollary implication that $dD=d\gamma$, so

$$\tan\gamma = 2 \tan i[1-(n+1)(\cos i/\mu \cos r)], \quad (1)$$

and therefore

$$\rho = a \sin i/\sin\gamma = a[\sin^2 i + \{\mu^2 \cos^2 i \cos^2 r\} / 4\{\mu \cos r - (n+1)\cos i\}^2]^{\frac{1}{2}}. \quad (2)$$

Equations (1) and (2), taken together with Snell's law, constitute parametric equations in ρ and γ for the caustic. These coordinates are not very convenient ones, but this is readily rectified (see below).

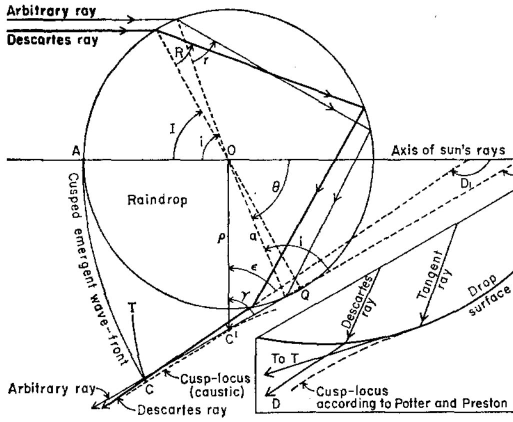
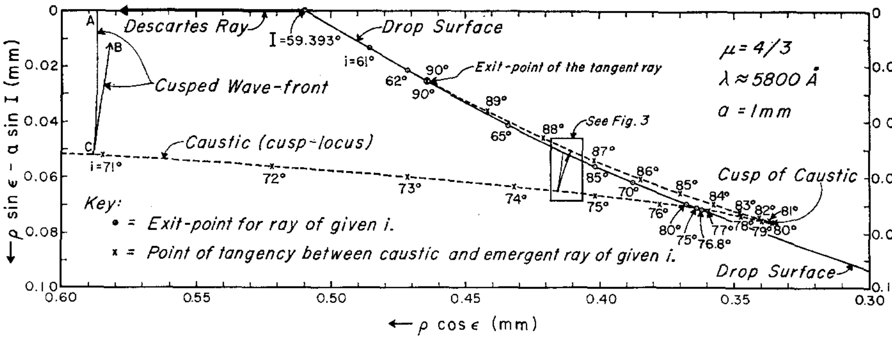


FIG. 1. Definition sketch for caustic geometry. For inset see text.

In the inset sketch at the lower right of Fig. 1 is displayed, in enlarged scale, the way in which Potter, and later Preston,[4] erroneously supposed the caustic to be jointly tangent to the drop and to that emergent ray which was incident at $i=90°$, approaching the contact-point smoothly *from the side of the Cartesian ray.* They drew the latter conclusion in what appears, at first, to be a valid manner: The caustic must touch the drop whenever $\rho=a$. From (2) and Fig. 1, one sees that $\rho=a$ in two cases. First, if $i=0$ then both $\sin i$ and $\sin\gamma$ approach zero in such a way that $\rho=a$. This specifies the surface-contact point of a second branch of the caustic, in which we are not here interested. Second, from (2), we have in the limit $i=90°$, $\rho=a$. And since the caustic, by definition, must be tangent to each emergent ray it meets, it follows that the caustic must touch the undersurface of the drop in such a way as to be tangent to the emergent tangentially incident ray at its exit point. Potter and Preston con-

FIG. 2. Correct geometry of the complete caustic near the drop.



[4] T. Preston, *Theory of Light* (MacMillan and Company Ltd., London, 1890).

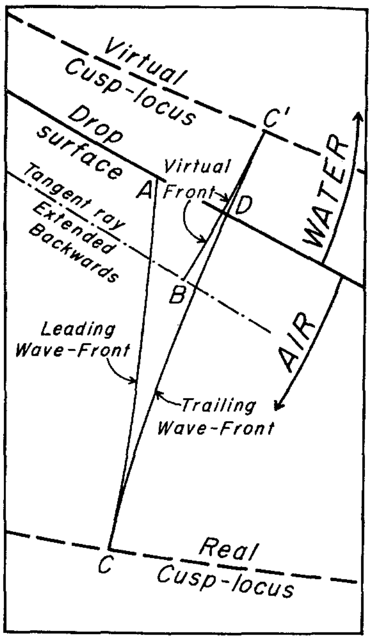


FIG. 3. Detailed view of complex doubly cusped wavefront in the neighborhood of the cusp of the caustic.

cluded from this that the caustic comes in to meet the drop in the manner sketched in the insert at the lower right of Fig. 1. That this conclusion is *incorrect* will now be shown.

Because γ is an angle subtended at the initially unknown location of the general point C', a first step must be some shift to more convenient coordinates than ρ and γ. The following choice proves convenient: Drop a perpendicular from O to O' (not shown in Fig. 1) on the backward extension of the Cartesian ray. Take O' as the origin of axes along and normal to the Cartesian ray. Then, from Fig. 1, plus considerations of elementary bow geometry, it follows that the coordinates of C' in this new system will be $\rho\cos\epsilon$ measured along the Cartesian ray in the direction towards the observer, and $\rho\sin\epsilon - a\sin I$ measured normal to the Cartesian ray, outward from the drop. Calling D_1 the deviation of the Cartesian ray, and defining $d = D - D_1$, we have $\epsilon = \gamma - d$.

Finding ordinary five-place-table accuracy inadequate in delineating precisely the caustic in the sensitive region near the drop surface (particularly near its curious cusp), an automatic computer affording eight-place accuracy was used to calculate values of the latter coordinates for the case of $\mu = 4/3$ (wavelength near 5800 Å), whence $I = 59.393°$. A drop radius of $a = 1$ mm was used, still small enough that the asphericity anomalies discussed by Volz[5] may be ignored. Computations were done at one-degree intervals from $i = 60°$ to $i = 90°$; and, for each i, the value of $\theta = \pi + i - 4r$ was also computed to locate the exit point of the ray.

In Fig. 2 is presented a detailed plot of the results on a scale adequate to reveal the true nature of the caustic near the raindrop. The solid curve is a part of the principal section of the drop, bounded at the upper edge by the Cartesian ray. The dashed curve is the caustic, the crosses marking the computed points at one-degree i-intervals. Locations of exit points are shown for the Cartesian ray and for enough other cases to depict the way in which θ varies with i.

The caustic, as correctly drawn in Fig. 2 is quite different from the previous conception as inset in Fig. 1. To be sure, the caustic *does* contact the drop surface at the exit point of the tangent ray and *does* contact it there tangentially, but this point is approached not from the side of the Cartesian ray but *from the far side after passing through an odd cusp lying within the drop*. As can be read from the figure, the cusp of the caustic occurs near $\theta = 80°$ for the specific case considered. The portion of the figure enclosed in a small rectangle is shown in enlarged version in Fig. 3. The leading branch AC is a real wavefront, as is also trailing portion CD; but DC' and $C'B$ are virtual, whence the upper branch of the cusped caustic (i.e., the branch within the drop) is a *virtual caustic*. In all, the geometry of the complete caustic is seen to be very different from the previous conception.

[5] F. E. Volz, in *Physics of Precipitation*, edited by H. Weickmann (American Geophysical Union, Washington D. C., 1960).