

CHAPTER XX.

Concerned with the motion of projectiles in air, which bodies are resisted in the square ratio of the speeds, where the body will not be acted on by the force of gravity from some given central position, as hitherto has been usually considered, but following the directions along the tangents at some points on a given line.

In the preceding chapter we have assigned the speed of a body acquired by falling along some given curve, or the amount of the speed remaining by ascending, as is apparent, according to uniform gravity acting along lines perpendicular to the horizontal, and by which the body may be acted on in air by a resistance in the square proportion to the speed. Truly in this chapter the motion of projectiles in air resisting in this manner remains to be examined, but with gravity varying, not only on account of the magnitude of the gravity itself acting, but also on account of the direction; which directions do not extend to a determined point, but we may suppose they are to be tangential to some curved line. However many forces have been expressed by the geometers in terms of all of these central forces, or as we are accustomed to call these, the action of central gravity either as it has been considered usually, from a given fixed central position, just as that itself called, and with these forces introduced and indicated above well enough. Now indeed we are going to extend the matter more general, we will assume the centre of that action to be changeable on one and the same curve, thus indeed, so that the moving body will be attracted to one and then another of the individual points in traversing the curve. In this manner all the centres put in place will be on a certain curved line, so that the actions of gravity are along the directions of the tangents. The problem of these forces, as I have said for the variable centres, may be seen, if not completely intractable, to be at first faced with difficulties. But, by applying the principles established in the first book, none the less we will set out a solution both simple and easy, from which henceforth all of the corollaries in place will be deduced, which actions observe the attributes directed not only towards a single point, but at least to several of these particular points.

602. And thus if AMm shall be the curve that the projectile M describes in air, leaving from the point A with a given speed along the direction of the tangent AC , with the

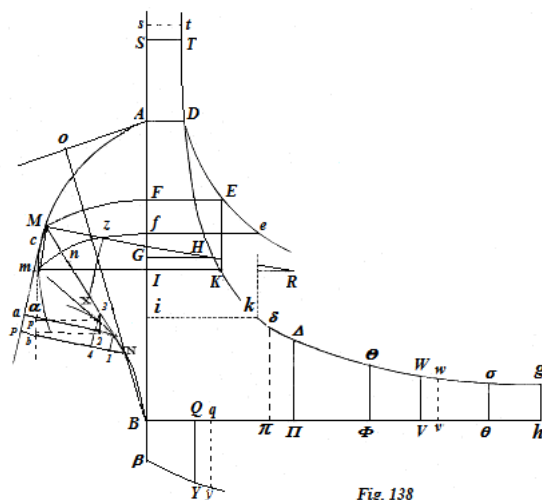


Fig. 138

resistance following the square ratio of the speed, evidently with the directions of the forces of gravity acting along AB , MN , mn , at different points everywhere, or considered to be acting from the centres B , N , n , and thus along the tangents at the prescribed points B , N , n , &c. of some curve BNn . Hence we will represent the forces acting on the body M directed towards the point N by MX [see enlarged diagram later], of which MN is taken in the direction, and with MR drawn normal to the curve AM at the point M , parallel to the tangent XZ of the curve MP through the point X at the said point of the curve, and XZ (§.118.) will express the tangential action of gravity at M .

603. Now the manner of such actions of gravity is required to be considered according to the case itself, where the body M on the curve AM is urged to be falling towards a number of diverse centres given in the positions ω , 2ω , 3ω , 4ω , &c. by the forces acting My , $M2y$, $M3y$, $M4y$, &c. For with the centre of gravity Ω present of the points y , $2y$, $3y$, $4y$, &c., the number of which shall be n , the resulting action from all the directions My , $M2y$, &c. (§. 49.) will be $n.M\Omega$, to which we shall make MX equal. This MX will be the force acting of the centre of gravity, by which the moving body will be affected at the point M of the curve, and the centre, to which this action MX is directed, will be the point N , at which the mean direction of all the forces My , $M2y$, $M3y$, $M4y$, &c. act on a certain curve Nn , (as hence the curve of the mean directions may be called). Therefore, in place of all the lateral forces acting My , $M2y$, &c. it will be agreed to consider the single mean action of these $MX = n.M\Omega$, so that indeed the whole matter may be reduced to the case of the preceding article.

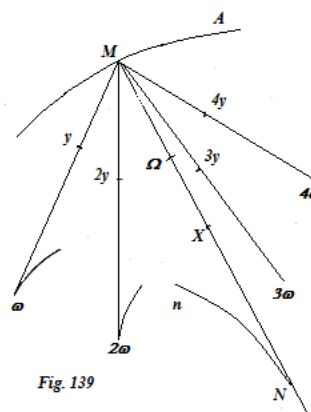


Fig. 139

HYPOTHESIS.

604. But with these it is required to assume, the resistance of the air to be exerted on the motion of the body M advancing along the curve AM, truly not expressed along the direction of gravity MN.

PROPOSITION LXXV. LEMMA.

605. *If on the hyperbola TKθ, between the asymptotes SB, BΦ, with the ordinates IK, GH, AD & ST drawn for the asymptote SB, and with the ordinates ΔΠ, ΘΦ, & WV drawn for the other asymptote BΦ; the quadrilinear form ΘΦVW will be equal to the excess, by which the quadrilinear form ΘΦΔΠ exceeds the two similar quadrilinear forms GHKI and ADTS, and the ratio BV to BΦ will be composed from the three ratios BΦ to BΠ, IB to GB, and AB to BS. That is, BV will be to BΦ, as the volume BΦ .IB.AB shall be to the volume BΠ .GB.BS.*

[It was well-established at the time, see e.g. Hutton's *Math. & Phil. Dictionary*, vol. 1, p. 619, that if successive distances marked out on the abscissa axis of a rectangular hyperbola shall form an increasing geometric progression, starting from the origin B: such as $BΠ, BΦ, Bh$; then in turn, those on the ordinate axis shall be in the inverse decreasing geometric progression, thus, so that the abscissas shall be in the constant proportion $BΦ:BΠ = Bh:BΦ$; consequently, the lengths of the corresponding ordinates $ΠΔ, ΦΘ, gh$, etc., shall be in an inverse decreasing sequence. The distances BI, BA, BS and their ordinates IK, AD, ST, etc., on the other axis may be related in the same manner. In which case the quadrilinear areas $ΠΦΘΔ, ΦhgΘ$, etc. are equal, with similar results for the vertical areas. Thus, the areas corresponding to successive terms of the geometric progression are themselves in an arithmetical progression, and are the natural logarithms of the ratios concerned.

We can express these areas in terms of the natural logs of the abscissas: thus, in an obvious notation for the horizontal axis, $ΠΦΘΔ = \log x_{\Phi} / x_{\Pi}$; $ΦhgΘ = \log x_h / x_{\Phi}$, etc. The areas and ratios in the proposition can then be determined in terms of logarithms.]

With Bh made the third proportional to $BΠ$ and $BΦ$, and with the ordinate gh erected, in addition, the quadrilinear forms are taken equal: $gσθh = GHIK$, then there will be $WVθσ = SADT$. Now, because

$$\begin{aligned}\Theta\Phi V W &= \Theta\Phi \Pi \Delta - GHIK - SADT \text{ (from the constr.)} \\ &= \Theta\Phi hg - \theta\sigma gh - SADT = \Theta\Phi \theta\sigma - SADT, \\ &= \Theta\Phi \theta\sigma - WV \theta\sigma, \\ &\text{there will be generally } WV \theta\sigma = SADT.\end{aligned}$$

And the ratio BV to BΦ is composed from the ratios BV to Bθ, Bθ to Bh, and Bh to BΦ, (or, as on account of the equal quadrilinear forms $WVθσ, SADT$; BV is to Bθ, as BA to BS, on account of $σθhg = GHIK$ there will be had $Bθ:Bh = BI:BG$, and finally on

account of $\Theta\Phi hg = \Theta\Phi\Pi\Delta$, there is $Bh : B\Phi = B\Phi : B\Pi$ from the ratios BA to BS, BI to BG and BΦ to BΠ; and thus $BV : B\Phi = B\Phi.BI.BA : B\Pi.BG.BS$. Q.E.D.

COROLLARY.

606. But, if the quadrilinear form $\Theta\Phi VW$ were equal to $\Theta\Phi\Pi\Delta - GHKI + ADTS$, there will be $BV : B\Phi = B\Phi.BI.BS : B\Pi.BG.BA$. The demonstration is almost the same.

PROPOSITION LXXVI. PROBLEM.

607. *If the moving body leaves in the direction AC with a given speed from the point A describes the curve AMm, and with the actions of the accelerations MX along the directions M, a tangent to some established curve BNn, with the resistance in air following the ratio composed from the ratios of the density of the medium and the square of the velocity of the moving body, to define the proportion of the speed of the body acquired at some point M of the curve to the initial speed at A, and the proportion of the accelerations acting at the same points of the curve M and A.*


$$\text{BI} = \text{MN}, \text{ Bi} = mn, \text{ B}\Phi = \text{BC}, \text{ B}\Pi = \text{NP} \text{ \& } \text{B}\pi = np,$$

and the ordinates may be drawn $IK, ik, \Phi\Theta, \Pi\Delta$ and $\pi\delta$ within the asymptotes AB, BQ of the hyperbola described through the point D . With all of these matters in place, there follows :

II. *Geometrical Analysis.* Since (§. 602.) the action ZX shall be derived from the central tangent MX , (§.481.) $ZX - R$ will be the acceleration of the body at the point M of the curve AM of the descent described, and $ZX + R$ the whole resistance of the moving body rising from the point M above MA : with the ascending case set aside for the present, we will examine the other case, which is the descent. Therefore the moment of the acceleration acting $ZX - R$ will be (§.484.) equal to the moment of the increasing speed BV , that is $(ZX - R).Mm = BV.Vu = ZX.Mm - R.Mm$, or with everything multiplied into the given rectangle BAD , there will be had
 $BA.AD.ZX.Mm - AB.AD.R.Mm = AB.AD.BV.Vu$, and therefore (following the hypothesis) $AB.AD.R = D.BV^2$ and $BAD = BVW$, there will be
 $AB.AD.ZX.Mm - D.BV^2.Mm = BV.WV.Vu$. And, because (§.604.) the air is not resisted by the actions of gravity, there will be had (§.154.) $BV^2 = MZ.MR$, therefore, by applying the preceding equation to this latter case, there will come about
 $AB.AD.ZX.Mm : MZ.MR, -QY.Qq : VW.Vu$. Since (by no. I of this section)
 $QY = D$ & $Qq = Mm$.

III. But, because the similar triangles Moa , BZX give $ZX : MZ = Ma : ao$, and the similar sectors $ca\alpha$, and RMm ; $Mm : MR = a\alpha : ac$ (and because, with the points M, m , joined together, the ratio Ma to ca is defined in the ratio of equality) $= a\alpha : Ma$, there becomes from the equation

$$ZX.Mm : MZ.RM = a\alpha : ao = a\alpha : PN(\text{const.}) = a\alpha : B\Pi,$$

therefore $AB.AD.ZX.Mm : MZ.MR = AB.AD : B\Pi$, that is (on account of all the equal rectangles in the hyperbola) $= \Pi\Delta.a\alpha$.

IV. Again there may be drawn through the point o the small line [*i.e.* incremental] ol parallel to the tangent MP , and through the point n the incremental lines $34, n2$ respectively parallel to the tangents MP, mp , and the points n, o, N , of the evanescent incremental lines connected together $o2, 14$, and $N3$ with the incremental arc Nn are defined in equal ratios, and likewise $o\alpha$ and ob . With which in place, because

$$\begin{aligned} NP - np(\text{constr.}) &= B\Pi - B\pi = \Pi\pi = PN - ao, + ao - \alpha o + bo(\text{or } \alpha o, \text{ equal to this}) \\ -pn &= N1 + a\pi + 14(\text{or } o2) = a\alpha + N4, \\ \text{there becomes } a\alpha &= \Pi\pi - N4, \& \Pi\Delta.a\alpha = \Pi\Delta.\Pi\pi - \Pi\Delta.N4. \end{aligned}$$

Indeed the similar triangles $N43$ and NPM , present

$$N4 : NP = N3 : NM = Nn : NM = Ii - Ff : NM.$$

For $mn + nN = MN - M\mu = BI - Ff$; therefore $nN = BI - Bi - Ff = Ii - Ff$. Hence, because $BI.IK = B\Pi.\Pi A$, there will be

$$\text{B}\Pi.\Pi\Delta.\text{N}4 : \text{B}\Pi, \text{vel NP} = \Pi\Delta.\text{N}4 = (\text{B}\text{I}.\text{I}\text{K}.\text{I}\text{i} - \text{B}\text{I}.\text{I}\text{K}.\text{F}\text{f}) : \text{B}\text{I} = \text{I}\text{K}.\text{I}\text{i} - \text{I}\text{K}.\text{F}\text{f}$$

(or with the parallel lines KE, *ke* drawn through the individual points K, *k* of the hyperbola parallel to the right line AB, with the lines FE, *fe* normal to the same AB, crossing at the points E, *e* of some curve DE) = $\text{I}\text{K}.\text{I}\text{i} - \text{F}\text{E}.\text{F}\text{f}$; there will be

$$\Pi\Delta.\alpha\alpha (= \Pi\Delta.\Pi\pi - \Pi\Delta.\text{N}4) = \Pi\Delta.\Pi\pi - \text{I}\text{K}.\text{I}\text{i} + \text{F}\text{E}.\text{E}\text{f}.$$

V. Hence since above (no. II. of this section) there shall be found

$\text{V}\text{W}.\text{V}\text{u} = \text{A}\text{B}.\text{A}\text{D}.\text{Z}\text{X}.\text{M}\text{m} : \text{M}\text{Z}.\text{M}\text{R} - \text{Q}\text{Y}.\text{Q}\text{q}$ (no. III) = $\Pi\Delta.\alpha\alpha - \text{Q}\text{Y}.\text{Q}\text{q}$; there will be (no. IV) $\text{W}\text{V}.\text{V}\text{u} = \Pi\Delta.\Pi\pi - \text{I}\text{K}.\text{I}\text{i} + \text{F}\text{E}.\text{F}\text{f} - \text{Q}\text{Y}.\text{Q}\text{q}$, and likewise everywhere; therefore everywhere the sum of all the

$$\text{W}\text{V}.\text{V}\text{u} = \text{sum } \Pi\Delta.\Pi\pi - \text{sum } \text{I}\text{K}.\text{I}\text{i} + \text{sum } \text{F}\text{E}.\text{F}\text{f} - \text{sum } \text{Q}\text{Y}.\text{Q}\text{q}.$$

That is,

$$\Theta\Phi\text{V}\text{W} = \Theta\Phi\Pi\Lambda - \text{A}\text{I}\text{K}\text{D} + \text{A}\text{F}\text{E}\text{D} - \text{B}\beta\text{Y}\text{Q};$$

for, if the point M is at A, all these areas likewise become zero. With the areas AFED and $\text{B}\beta\text{Y}\text{Q}$ taken equal to the quadrilinear hyperbolic areas AGHD & ASTD, and there will be $\Theta\Phi\text{V}\text{W} = \Theta\Phi\Pi\Lambda - \text{G}\text{I}\text{K}\text{D} - \text{A}\text{S}\text{T}\text{D}$, and consequently (§.605).

$\text{B}\text{V} : \text{B}\Phi = \text{B}\Phi.\text{B}\text{I}.\text{B}\text{A} : \text{B}\Pi.\text{G}\text{B}.\text{S}\text{B}$ (or with BC, MN, NP put in place of BΦ, BI, & BΠ) = $\text{B}\text{C}.\text{M}\text{N}.\text{A}\text{B} : \text{N}\text{P}.\text{G}\text{B}.\text{S}\text{B}$. Now, because

$\text{B}\text{C} : \text{A}\text{B} = \text{sine of the angle BAC} : \text{to the radius or the whole sine}$, and

$\text{N}\text{P} : \text{M}\text{N} = \sin(\text{NMP}) : \text{rad.}$, there will be $\text{B}\text{V} : \text{B}\Phi$, that is, the speed of the moving body at M to the initial speed at A, just as AB^2 by $\sin(\text{BAC})$ to $\text{G}\text{B}.\text{S}\text{B}$ by $\sin(\text{NMP})$. Thence, if $v\text{A}$ indicates the speed at A, there will be BV, or the speed at M, as also expressed by

$$v\text{M} = \text{A}\text{B}^2.v\text{A}.\sin(\text{BAC}) : \text{G}\text{B}.\text{S}\text{B}.\sin(\text{NMP}).$$

VI. Again, because $\text{M}\text{X}.\text{M}\text{R} = \text{M}\text{Z} : \text{M}\text{R}$, or,

$$\text{M}\text{X} : \text{M}\text{Z} = \text{rad} : \sin(\text{MXZ}) \text{ or } \sin(\text{NMP}), \text{ and } (§.154) \text{M}\text{Z}.\text{M}\text{R} = v\text{M}^2;$$

from the equation there becomes $\text{M}\text{X}.\text{M}\text{R} : v\text{M}^2 = \text{rad.} : \sin(\text{NMP})$, and (no. V.)

$$v\text{M}^2 : v\text{A}^2 = \text{A}\text{B}^4.\sin^2(\text{BAC}) : \text{G}\text{B}^2 : \text{S}\text{B}^2.\sin^2(\text{NMP});$$

and (by putting AR and A for the radius of osculation, and with the initial motion on the curve from the point A from the central action A) $v\text{A}^2 : \text{A}.\text{A}\text{R} = \sin(\text{BAC}) : \text{rad.}$ Hence from the equation, and by the combining of the ratios :

$$\text{M}\text{X}.\text{M}\text{R} : \text{A}.\text{A}\text{R} = \text{A}\text{B}^4.\sin^3(\text{BAC}) : \text{S}\text{B}^2.\text{G}\text{B}^2.\sin^3(\text{NMP}).$$

A certain magnitude M is assumed, of such a size, that M.MR shall be to A.AR, just as $\text{A}\text{B}^2.\sin^3(\text{BAC})$ to $\text{G}\text{B}^2.\sin^3(\text{NMP})$, or,

$$\text{M} = \text{A}.\text{A}\text{R}.\text{A}\text{B}^2.\sin^3(\text{BAC}) : \text{M}\text{R}.\text{G}\text{B}^2.\sin^3(\text{NMP}),$$

and it will be found that $\text{M}\text{X} : \text{M} = \text{A}\text{B}^2 : \text{S}\text{B}^2$, that is, the action centre for air resistance is at some point M on the curve, = $\text{M}.\text{A}\text{B}^2 : \text{S}\text{B}^2$, evidently with the body completing the

arc of the curve AM from A on descending, or for the act of the curve nNB it will describe by ascending.

VII. But if indeed it may ascend on the arc MA from M towards A, that is be receding from the curve nNB , in this case there will be found $\Theta\Phi VW = \Theta\Phi\Pi\Delta - GIKH + ASTD$, and thus, as by paragraph 606, and which have been shown in a number of those preceding, $MX = M.SB^2 : AB^2$. Which were all required to be found.

608. Moreover it is required to be known that the letter $\int$ indicated in this proposition is to be the known characteristic of the sines of these angles, to which it is prefixed, and $\int \overline{BAC^3}$, $\int \overline{NMP^3}$ signify the cubes of the sines of the angles BAC, NMP, and with the remaining sign, evidently A, to denote the central action of gravity at the initial point of the curve A, & AR (as not to be expressed in the figure in order to avoid confusion) the radius of osculation of the circle, or of the same curvature as the curve at the same point A. Indeed the ratio SB to AB, in which SB is put always greater than AB, is the same as with the ratio of the abscissas of any quadrilinear form of the hyperbola ASTD (by constr.), of the equal area B β YQ of the graph of the density of the air. On that account the terms of the ratio SB to AB, are not restricted to a determined magnitude, but, for the case concerned, can be taken to be larger or smaller, as long as the ratio between these terms may remain equal to that between the abscissas of any of the aforementioned equal areas of the hyperbolic trapeziums in the graph of the air density.

[In order not to confuse the sign $\int$ with that for integration, and to render the work easier to follow, we have replaced this sign everywhere by the sin, as in modern usage. The Latin original attached preserves the original notation, if you want to examine it.]

COROLLARY I.

609. And thus in a medium of zero density, that is, in a vacuum the areas B β YQ or ASTD vanish, and thus the ratio SB to AB will be changed into a ratio of equality, and there becomes

$$vM : vA = AB.\sin(BAC) : GB.\sin(NMP). \text{ And } MX (= M.AB^2 : SB^2, \text{ or } M.SB^2 : AB^2) = M.$$

Therefore M, the value of which has now been determined (§. 607. no. VI.) ; expresses the most general action of centre gravity *in vacuo*.

COROLLARY II.

610. If the moving body M may be acted on simultaneously by some number of lateral forces, M_1y , M_2y , M_3y , M_4y , &c. according to the different centres ω , 2ω , 3ω , 4ω , &c. ; the mean direction MN of these, which (§.603 and 49.) will pass through the centre of gravity Ω of all the points y , $2y$, $3y$, $4y$, &c., will touch some curve nN , that hence is accustomed to be called the curve of the mean direction of the lateral forces acting. Therefore, because (§.603.) $MX = n.M\Omega$

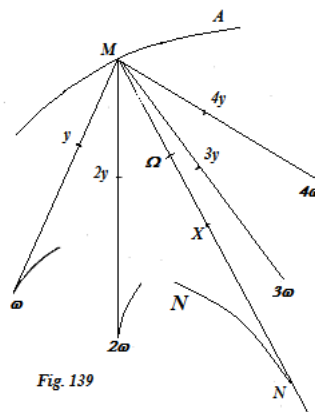
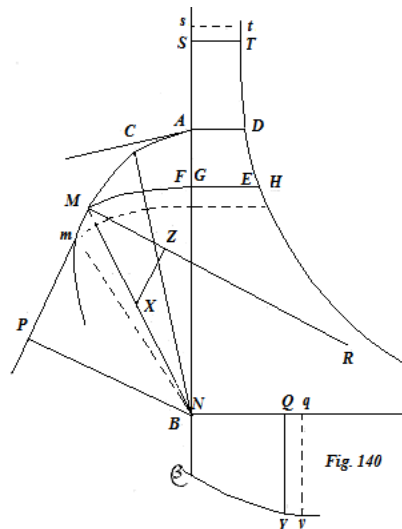


Fig. 139

611. Because (§.607. no. I.) we have assumed $AB.AD.R = D.BV^2$, and (§.154.) $BV = MZ.MR$, there will be $R : MZ = D.MR : AB.AD$. And $MZ : MX = NP : NM = \sin(NMP) : \text{rad.}$ therefore from the equation, the resistance R to the central action $MX = D.MR.\sin(NMP) : AB.AD.\text{radius}$. Likewise because $MX : ZX = MN : MP = \text{rad.} : \sin(MNP)$, again there will be from the equation the resistance R to the tangential action of gravity

612. If the actions of gravity MX are directed to a centre at a given position B, which is the case of these actions or centripetal forces, such as has been accustomed to be considered until this time by geometers, the whole curve NB being contracted into the point B, and of which the evolutes MF, $m\mu f$ will become concentric circular arcs described from the centre B, therefore the individual NM, BI & BF, and indeed GB, will become equal, then with the curve DE coincident with the respective hyperbolic arc DK. And thus the formula of number VI. §.607. also will be changed into

$vA^2 = A.BC.AR : AB$. This last formula for the central actions *in vacuo*, and with only one centre of attraction present, agrees well with that if, as above (§.156.) I have given a demonstration, and likewise I have referred to the citations in place by the most distinguished geometers Johan Bernoulli, Abr. de Moivre, and Guido Grandi being set out in similar rules, but by another way. For in Fig. 37, the lines $N\alpha$, AI , DR , DN , nZ , Dq , signify the same, and in Figure 138. the letters A , vA , and the lines BC , NM , MR & NP respectively.



SCHOLIUM I.

613. Now the curve AM shall be a logarithmic spiral, in which the actions of gravity MX shall be directed towards the pole B, thus so that the points N, B shall be one and the same, from which, because the individual angles BAC, NMP are equal, the ratio of the lines MP, MN & NP will be given everywhere. The density of the air at M may be put as $h : MP : MN^2$, where h indicates some number, and thus there will be, on account of the given ratio, MP to MN inversely, as MN, the distance of the moving body from the centre N or B; and with that in place, $D = h.AB.AD.MP : MN^2$, there will be found $D.Mm = QY.Qq = ST.Ss = h.IK. li$, and consequently $ASTD = h.AIKD$, and therefore

$$AB : SB = BI^h : AB^h = MN^h : AB^h, \text{ \& } AB^2 : SB^2 = MN^{2h} : AB^{2h}.$$

Again, because the individual angles BAC, NMP, &c. are equal and as a consequence the sines of these, and the radii of osculation AR, MR at A, M with the distances AB, MN proportional, there will be now (§.612.)

$$M : A = \left(AR.AB^2 . \sin^3(\overline{BAC}) : MR.MN^2 . \sin^3(\overline{NMP}) \right) = AB^3 . MN^3. \text{ And (§.607. no. VI.)}$$

there was $MX : M \left(= AB^2 : BS^2 \right) = MN^{2h} : AB^{2h}$; therefore from the equation,

$MX : A = MN^{2h-3} : AB^{2h-3}$. Therefore the centripetal force, that is, the action of the central MX everywhere on the logarithmic spiral, is as MN^{2h-3} , if the body by descending will describe the arc AM; and conversely, as MN^{2h+3} by ascending it approaches through M towards A upon the arc MA.

Hence, because generally there is (§.611.) $R : MX = D.MR.\sin(NMP) : AB.AD.\text{rad.}$ and D (following the hypothesis) $= h.AB.AD.MP : MN^2$; there will be

$R : MX = h.MR.MP.\sin(NMP) : MN^2.\text{rad.}$ (or, because NP is to NM, as $\sin(NMP) : \text{rad.}$ and the rectangle NP.MR equals MN^2) $= h.MP : MN$, that is, the ratio R to MX will be constant everywhere or given.

Therefore, if $h = \frac{1}{2}$, MX will be inversely in the square ratio of the distance MN, and the resistance R to the central action MX, as $\frac{1}{2}$ MP to MN.

But if indeed h where $= 1 - \frac{1}{2}n$, MX would be found inversely as

$$MN^{n+1} \text{ and } R : MX = \left(1 - \frac{1}{2}n \right).MP : MN.$$

Therefore the density at M, shall be as the tangent of the angle AFM. Therefore either $D = 3AD \cdot \text{rad.} : 2 \cdot \text{tang. MFE}$, or $NMP = 3 \cdot AF : AD \cdot \text{rad.} : 2AF \cdot \text{tang. NMP}$.

Indeed there is $R : ZX$ (§. 611.) $= D \cdot MR \cdot \text{tang. NMP} : AB \cdot AD \cdot \text{rad.}$ (or in this case, where AF being assumed in place of AB in Fig. 138.) $= D \cdot AF \cdot \text{tang. NMP} : AF \cdot AD \cdot \text{rad.}$ (or with the value of D substituted) $= 3AF \cdot \text{tang. NMP} : 2AF \cdot \text{tang. NMP} = 3 : 2$. Therefore also

$$R : MX = 3 \cdot \sin(MFA) : 2 \cdot \text{rad.} = 3 \cdot GH : 2MF.$$

Which were all found by Bernoulli (Act. Lips. 1713. Febr. pag. 93.) and are in partial agreement with the works of Newton, even if deduced from another basis. I talk about the descending case, since the ascending case is impossible, as observed elegantly by the illustrious Newton; and indeed it may happen in this case the resistance to the weight, to be as $-3GH$ to $2MF$, that is, as a negative magnitude to a positive one.

COROLLARY VII.

617. Corollary VI. also provides for us the foundations of finding the construction of the curve of a projectile travelling in air of a uniform density, but with the resistance in the square ratio of the speed of the moving body, clearly so that there with the gravity in place uniform and by acting along its presumed parallel directions. For with a certain Q provided :

$$Q = \int -2dm\sqrt{1+mm} : m^3, \text{ there will be } y = AF + \int dm : m^3Q, \text{ and } FG = x = \int dm : mmQ.$$

SCHOLIUM III.

618. We have shown the solution, which according to a particular hypothesis of the air resistance following a ratio composed from the density and the square of the velocity, the problem can be produced more generally for a ratio composed from the ratio of the density and some multiple or sub-multiple of the speed of the body and that with so little work, with the geometry of the preceding problem proposed analytically, so that it can be done with no change in that, except only on the curve $B\beta$, which in the more general hypothesis is no further a graph of the density of the air, but a curve of another kind, concerning which a little is required to be shown.

619. The curve βY shall be of this kind, so that everywhere with the existence of its abscissa BQ equal to the arc of the curve AM, its ordinate QY shall be to the air resistance R at the point M on the curve, thus as the given rectangle AB.AD to BV^2 the square of the velocity acquired at M, and this curve B β generally is required to be taken in place of the graph of the density of the air used before in the analysis of the problem,

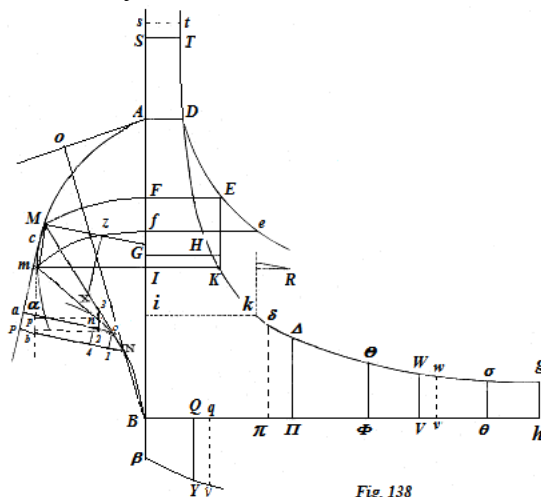


Fig. 138

and the rest are required to be resolved in a straight forwards manner as in §. 607, clearly by making the area B β YQ of the new curve βY equal to that of the four sided hyperbolic figure ASTD ; and now also the speed of the body acquired at M, or

$vM = AB^2 \cdot vA \cdot \sin(BAC) : GB \cdot SB \cdot \sin(NMP)$, as was being found in a straight forwards manner above (§. 607. no. V.), only with this distinction that now SB is going to be sought of another magnitude than what is sought up to the present. For brevity it is assumed that $V = AB \cdot vA \cdot \sin. BAC : GB \cdot \sin.NMP$, and the velocity V, which the moving body would have acquired *in vacuo* by descending on the arc of the curve AM at its extremity M, and vM or $BV = AB \cdot V : SB$ in the case of the descent, or $BV = SB \cdot V : AB$ in the case of ascending.

620. Now, because (by constr.) B β YQ = ASTD, and B β yq = AstD, there will be

$QY \cdot Qq = ST \cdot Ss$, but $QY = AB \cdot AD \cdot R : BV^2$ (or by requiring the value $AB \cdot V : SB$, of BV) = $AD \cdot R \cdot SB^2 : AB \cdot V^2$, and $Qq : Mm$, there will be $AD \cdot R \cdot SB^2 \cdot Mm : AB \cdot V^2 = ST \cdot Ss$, or with everything being multiplied by SB,

$$AD \cdot R \cdot SB^3 \cdot Mm : AB \cdot V^2 = SB \cdot ST \cdot Ss = AB \cdot AD \cdot Ss,$$

and thus hence it will be elicited that $R \cdot Mm : V^2 = AB^2 \cdot Ss : SB^3$. And thus

$(R \cdot Mm : V^2) =$ to the sum of all $(AB^3 \cdot Ss : SB^3)$ that is $= (SB^2 - AB^2) : 2SB^2$, and thus

on designating all the ratios $(R \cdot Mm : V^2)$ by $\int R \cdot Mm : V^2$, there will be

$2 \int R \cdot Mm : V^2 = (SB^2 - AB^2) : SB^2$. Hence $SB^2 - AB^2 = 1 : 1 - 2 \int R \cdot Mm : V^2$; for the descending case.

621. For the case of ascending in place of BV (§. 619.) it is necessary to substitute SB.V : AB into QY, which for each case is = AB. AD.R : BV², and there becomes QY.Qq = AB³.AD.R.Mm : SB².V² = ST.Ss, or by multiplying in turn the individual terms by SB; AB³.AD. R. Mm : SB. V² = SB.ST.Ss = AB.BD.Ss, hence on dividing by AB.AD, there will be found AB².R.Mm.V² = SB.Ss. Therefore all the AB². R.Mm : V² = to the sum of all the SB.Ss = $\frac{1}{2}SB^2 - \frac{1}{2}AB^2$, therefore $2AB^2 \cdot \int R.Mm : V^2 = SB^2 - AB^2$. And thus $SB^2 : AB^2 = 1 + 2 \int R.Mm : V^2$, in the case of ascending.

These are the two canons for any kind of resistance, whither or not that resistance may depend on the velocities and on the density of the air.

622. But, if the resistance shall be in a ratio composed from the density of the air and from the ratio of the speed in some multiple of the speed, that is, if R shall be as D.BV^h, where h is a number assumed as it pleases. Or, taken according to the demands of homogeneity, there shall be R = D.BV^h : AB^h. Nevertheless, because (§.620.)

$AB^2 : SB^2 = 1 - 2 \int R.Mm : V^2$ for the descending case, there will be with the element of the amount taken on both sides [*i.e.* differentiate], $-2AB^2.Ss : SB^3 = 2R.Mm : V^2$, or (by substituting the value of R and dividing the equality by -2) there will emerge $AB^2.Ss : SB^3 = V^{h-2}.D.Mm : SB^h$, and thus $SB^{h-3}.Ss = V^{h-2}.D.Mm : AB^2$. Therefore all the terms $V^{h-2}.D.Mm : AB^2$, that is, $\int V^{h-2}.D.Mm : AB^2 =$ all the terms $SB^{h-3}.Ss$, that is, $= \frac{1}{h-2}SB^{h-2} - \frac{1}{h-2}AB^{h-2}$; and thus hence the canon is elicited

$SB = \sqrt[h-2]{(AB^{h-2} + h-2 \int V^{h-2}.D.Mm : AB^2)}$, for the descent of the body on the curve AM.

In a similar manner there will be found :

$SB = AB^2 : \sqrt[h-2]{(AB^{h-2} + 2-h \int V^{h-2}.D.Mm : AB^2)}$ for the case of the ascent, when the moving body advances from M towards A.

613. Therefore as everything in the solution of this problem have been scattered around, everything being produced must be viewed together, no. V. §. 607, produces the speed at M = AB.V : SB and SB.V : AB the one for the ascending case and the other for the body descending, where V is (§. 619.) $AB.v \sin(BAC)$ applied to $GB \sin(NMP)$, that is, the expression of the speed of the moving body at the point M of the curve carried *in vacuo*. Therefore the speed acquired by the body in air at some point M of the curve to the speed of the same *in vacuo* at the same point of the curve, shall be as AB to SB; and the speed remaining of the body at M for the resisting medium to the speed *in vacuo*, as SB to AB, when it ascends on the curve. And no. VI. §.607. The central action in air to the action of gravity in vacuo with the body moving in each case at some point M of the curve *in*

vacuo, as AB^2 to SB^2 , when the body by descending describes the curve AM, or, thus as SB^2 to AB^2 , if by ascending, by going through M towards A. Truly these magnitudes V and M, evidently the speed *in vacuo* and the central action with respect to the point M of the curve, now have been found most generally, clearly so that as a little above

$V = AB.vA.\sin(BAC):GB.\sin(NMP)$, and $M = A.AR.\sin^3(BAC):MR.\sin^3(NMP)$, in which formula GB, with the agreement are given always with the quadratures of the curvilinear figures. Truly these SB in paragraphs 620, 621 and 622 have been shown more generally. From our most general solution innumerable other corollaries may be able to be deduced, which I have omitted for the sake of brevity.

CAPUT XX.

De motu projectorum in aëre, qui missili in duplicata ratione celeritatum resistit, cum corpus a sollicitationibus gravitatis non ad aliquod centrum positione datum, ut hactenus considerari solebant, tendentibus, urgetur, sed secundum directiones lineam quamcunque curvam positione datam contingentes.

In præcedenti Capite celeritates mobili in curva quacunque cadenti acquisitas, vel ascendenti residuas, assignavimus, cum scilicet, a gravitate uniformi recta ad horizontem tendente urgetur in aëre secundum duplicatam proportionem celeritatum resistente. In hoc vero Capite examinandi restant motus projectilium in ejus Modi aëre resistente; sed pro gravitate variante, tam ratione ipsarum gravitatis sollicitationum, quam ratione directionum; quas directiones ad nullum determinatum punctum tendere, sed quamcunque lineam curvam contingere supponemus. Ab omnibus, quotquot de viribus centralibus scripserunt, geometris, harum virium centralium, vel ut nos eas vocare solemus, sollicitationum gravitatis centralium meta vel centrum positione datum & immutabile, considerari consuevit, veluti id nomen ipsum hisce viribus inditum satis superque declarat. Nos vero rem generalissime pertractaturi, sollicitationum illarum centrum in una eademque curva mutabile assumemus, ita quidem, ut mobile in singulis curvæ percurrendæ punctis ad aliud atque aliud centrum sollicitationum urgeatur. Hoc modo centra omnia posita erunt in quadam linea curva, quam sollicitationum gravitatis directiones contingunt. Problema harum virium, pro centris, quæ dixi, variabilibus, prima fronte difficillimum, si non prorsus intractabile, videtur. Veruntamen, principiis in primo libro jam expositis insistentes, solutionem non minus simplicem quam facilem trademus, ex qua deinceps omnia instar corollariorum deducuntur, quæ sollicitationes ad unicum tantum centrum directas respiciunt symptomata, aut saltem eorum præcipua.

601. Adeoque, si AMm sit curva, quam projectile M data cum celeritate exiens ex

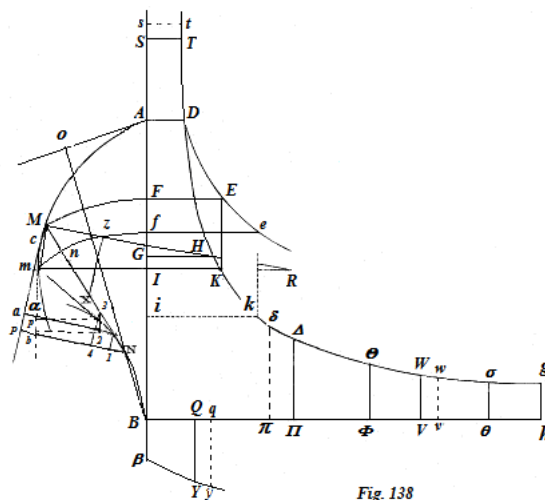
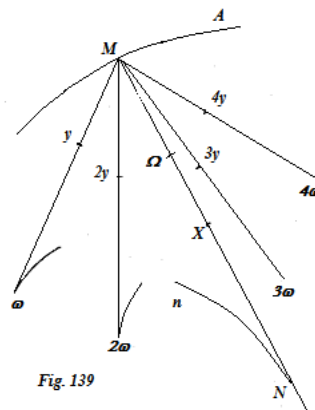


Fig. 138

puncto A secundum directionem tangentis AC describat in aëre secundum rationem duplicatam ipsi resistente, cum scilicet directiones sollicitationum gravitatis AB , MN ,

mn , ubique ad diversa puncta seu centra B , N , n respiciunt, atque adeo curvam quamcunque BNn in præfatis punctis B , N , n , &c. contingunt. Solicitationem mobilis M ad punctum N directam deinceps repræsentabimus per MX , in ejus directione MN sumptam, ductaque MR normali ad curvam AM in puncto M , & per X parallela XZ tangenti curvæ MP in dicto curvæ puncto, exponetque XZ (§.118.) sollicitationem gravitatis tangentialem in M .

603. Talis sollicitationes gravitatis considerandi modus jam continet sub se casum, quo corpus M in curva AM in cedens ad plura diversa centra positione data ω , 2ω , 3ω , 4ω , &c. urgetur sollicitationibus My , $M2y$, $M3y$, $M4y$, &c. Nam existente Ω centro gravitatis punctorum y , $2y$, $3y$, $4y$, &c. quorum numerus sit n , sollicitatio ex omnibus lateralibus My , $M2y$, &c. resultans (§. 49.) erit $n.M\Omega$, cui æqualem faciamus MX . Hæc MX erit sollicitatio gravitatis centralis, qua mobile in puncto curvæ M afficietur, & centrum, ad quod hæc sollicitatio MX dirigitur, erit punctum N , in quo media directio omnium My , $M2y$, $M3y$, $M4y$, &c. curvam quandam Nn , (quam proinde curvam mediarum directionum nominare licet) contingit. Propterea, loco omnium illarum sollicitationum lateralium My , $M2y$, &c. unicam illarum mediam sollicitationem $MX = n.M\Omega$ considerare convenit, ut adeo tota res reducatur ad casum articuli præcedentis.



HYPOTHESIS.

604. In hisce autem supponendum, resistantiam aëris tantum exseri in motum corporis M in curva AM incedentis, non vero gravitatis impressiones juxta directiones MN .

PROPOSITIO LXXV. LEMMA.

605. Si in hyperbola $TK\theta$, inter asymptotas SB , $B\Phi$, ductis ordinatis IK , GH , AD & ST ad asymptotam SB , ad alteram vero $B\Phi$, ordinatis $\Delta\Pi$, $\Theta\Phi$, & WV ; quadrilineum $\Theta\Phi VW$ æquaverit excessum, quo quadrilineum $\Theta\Phi\Delta\Pi$ superat duo simul quadrilinea $GHIK$ & $ADTS$, ratio BV ad $B\Phi$ componetur ex tribus rationibus $B\Phi$ ad $B\Pi$, IB ad GB , & AB ad BS . Hoc est, erit BV ad $B\Phi$, ut sit $B\Phi .IB.AB$, ad solidum ex $B\Pi .GB.BS$.

Facta Bh tertia proportionali ad $B\Pi$ & $B\Phi$, erectaque ordinata gh , sit insuper quadrilineum $g\sigma\theta h = GHIK$, eritque $WV\theta\sigma = SADT$. Nam, quia

$$\begin{aligned}\Theta\Phi VW &= \Theta\Phi\Pi\Delta - GHIK - SADT \text{ (constr.)} \\ &= \Theta\Phi hg - \theta\sigma gh - SADT = \Theta\Phi\theta\sigma - SADT, \\ &= \Theta\Phi\theta\sigma - WV\theta, \text{ erit omnino } WV\theta\sigma = SADT.\end{aligned}$$

Atqui ratio BV ad BΦ componitur ex rationibus BV ad Bθ, Bθ ad Bh, & Bh ad BΦ, (seu, quia propter quadrilinea æqualia WVθσ, SADT; BV est ad Bθ, ut BA ad BS, propter σθhg = GHK habetur Bθ : Bh = BI : BG, ac denique propter ΘΦhg = ΘΦΠΔ, est Bh : BΦ = BΦ : BΠ) ex rationibus BA ad BS, BI ad BG & BΦ ad BΠ; atque adeo BV : BΦ = BΦ.BI.BA : BΠ.BG.BS. Quod erat demonstrandum.

COROLLARIUM.

606. Sed, si quadrilineum ΘΦVW fuerit æquale ΘΦΠΔ – GHKI + ADTS, erit BV : BΦ = BΦ.BI.BS : BΠ.BG.BA. Demonstratio eadem ferme est.

PROPOSITIO LXXVI. PROBLEMA.

607. Si mobile in directione AC data cum celeritate ex puncto A exiens, & sollicitationibus acceleratricibus MX secundum directiones M, curvam quamcunque BNn contingentes, citatum, in aëre secundum rationem compositam ex rationibus densitatum medii & duplicatae velocitatum mobilis resistente, curvam AMm describat, definire proportionem celeritatis mobilis in quolibet curvæ puncto M acquisitæ ad celeritatem initialem in A, ac proportionem sollicitationem acceleratricium in iisdem curvæ punctis M & A.

I. Sunto Mm elementum curvæ AM, per cujus elementi terminos transeant tangentes MN, mn curvæ BNn sese intersecantes in puncto o; MP & mp tangentes curvæ AMm in punctis M, m convenientes in c, super quas tangentes demittantur ex punctis N, θ & n normales NP, oa (tangente interiorē cb secans in a) ob & np, ac denique MR, mR perpendiculares curvæ AM in punctis M & m, adeo ut, coeuntibus his punctis, hæ perpendiculares curvæ futuræ sint radius circuli osculatoris curvi in puncto M vel m, seu circuli æque curvi in his punctis. Sint porro curvæ MF, muf evolutæ arcuum BN, BNn; rectæ vero BΦ & BV in linea indefinita BQ ipsi BA normali abscissæ exponant velocitates mobilis in punctis A & M, & Vu elementum ipsius BV; extensaque curva AM in rectam BQ erectaque QY perpendiculari ad BQ ac densitati aëris in omni curvæ AM termino proportionali, orietur figura; vel potius curva βY, quam scalam densitatum aëris deinceps vocabo. Denique assumtis D & R pro nominibus, densitatis & resistentiæ aëris, ponatur ubique R ut D.BV², vel, ad complenda homogenea, adscito AB.AD rectangulo dato, sit ubique AB.AD.R = D.BV². Ac fiant

$$BI = MN, Bi = mn, B\Phi = BC, B\Pi = NP \text{ \& } B\pi = np,$$

ducanturque ordinatæ IK, ik, ΦΘ, ΠΔ ac πδ hyperbola intra asymptotas AB, BQ per punctum D descripta. Hisce omnibus præparatis sequitur.

II. *Analysis Geometrica.* Cum (§. 602.) ZX sollicitatio sit tangentialis ex centrali MX. derivata, erit §.481.) ZX – R sollicitatio acceleratrix mobilis in puncto M curvæ AM descensu descriptæ, & ZX + R resistantia totalis mobilis ex puncto M super MA ascendenti : sed seposito tantisper casu ascensionis, examinabimus alterum, qui est descensus. Erit igitur (§.484.) momentum sollicitationis acceleratricis ZX – R æquale momento celeritatis crescentis BV, id est (ZX – R).Mm = BV.Vu = ZX.Mm – R.Mm, vel

ducendo omnia in datum rec-lum BAD, habetur

BA.AD.ZX.Mm – AB.AD.R.Mm = AB.AD.BV.Vu, & quia (secundum hypothesin)

AB.AD.R = D.BV² ac BAD = BVW, erit AB.AD.ZX.Mm – D.BV².Mm = BV.WV.Vu.

Et, quia (§.604.) aër gravitatis sollicitationibus non resistit, habetur (§.154.)

BV² = MZ.MR, idcirco, applicando æqualitatem præcedentem ad hanc postremam, proveniet AB.AD.ZX.Mm : MZ.MR, – QY.Qq : VW.Vu. Cum (num. 1, hujus.)

QY = D & Qq = Mm.

III. At, quia triacula similia Moa, BZX præbent ZX : MZ = Ma : ao, & sectores similes caa, ac RMm; Mm : MR = aa : ac (& quia, coeuntibus punctis M, m, ratio Ma ad ca in rationem æqualitatis definit) aa : Ma erit ex æquo

$$ZX.Mm : MZ.RM = aa : ao = aa : PN (\text{const.}) = aa : B\Pi,$$

ergo AB.AD.ZX.Mm : MZ.MR = AB.AD : B\Pi, hoc est (propter omnia rectangula in hyperbola æqualia) = \Pi\Delta.aa.

IV. Ducantur porro per punctum o lineola o\Phi parallela tangenti MP, & per punctum n lineolæ 34, n2 parallelæ respective tangentibus MP, mp, & coeuntibus punctis n; o, N, lineolæ evanescentes O2, 14, & N3 arcu Nn in rationes æqualitatis definient, perinde ac oa & ob. Quo posito, quia

$$NP - np (\text{constr.}) = B\Pi - B\pi = \Pi\pi = PN - ao, + ao - \alpha o + bo (\text{seu } \alpha o, \text{ ipsi æqualis})$$

$$-pn = N1 + a\pi + 14 (\text{vel } o2) = \alpha\alpha + N4, \text{erit } \alpha\alpha = \Pi\pi - N4, \& \Pi\Delta.\alpha\alpha = \Pi\Delta.\Pi\pi - \Pi\Delta.N4.$$

Triacula vero similia N43 & NPM, præbent

N4 : NP = N3 : NM = Nn : NM = Ii - Ff : NM. Nam mn + nN = MN - M\mu = BI - Ff; ergo nN = BI - Bi - Ff = Ii - Ff Hinc, quia BI.IK = B\Pi.\Pi A, erit

$$B\Pi.\Pi\Delta.N4 : B\Pi, \text{vel } NP = \Pi\Delta.N4 = (BI.IK.Ii - BI.IK.Ff) : BI = IK.Ii - IK.Ff$$

(seu ductis per singula hyperbolæ puncta K, k parallelis KE, ke rectæ AB, lineis FE, fe eidem AB. Normalibus, occurrentes in punctis E, e alicujus curvæ DE) = IK.Ii - FE.Ff; erit

$$\Pi\Delta.\alpha\alpha (= \Pi\Delta.\Pi\pi - \Pi\Delta.N4) = \Pi\Delta.\Pi\pi - IK.Ii + FE.Ef.$$

V. Hinc cum supra (num. II. hujus) reperta sit

VW.Vu = AB.AD. ZX.Mm : MZ.MR – QY.Qq (num. III hujus) = \Pi\Delta.\alpha\alpha – QY.Qq; erit (num.IV) WV.Vu = \Pi\Delta.\Pi\pi – IK.Ii + FE.Ff – QY.Qq, & sic ubique; ergo

omnia WV.Vu = omn. \Pi\Delta.\Pi\pi – omn. IK.Ii + omn. FE.Ff – omn. QY.Qq. Hoc

est, \Theta\Phi VW = \Theta\Phi\Pi A – AIKD + AFED – B\beta YQ; nam, si punctum M est in A, omnes hæ areæ simul a nihilo incipient. Areis AFED & B\beta YQ capiantur æqualia quadrilinea hyperbolica AGHD & ASTD, eritque \Theta\Phi VW = \Theta\Phi\Pi A – GIKD – ASTD, ac per consequens (§.605). BV : B\Phi = B\Phi.BI.BA : B\Pi.GB.SB (vel restitutis BC, MN, NP loco B\Phi, BI, & B\Pi) = BC.MN.AB : NP.GB.SB. Jam, quia BC : AB = sinus anguli BAC :

radium vel sinum totum, & NP : MN = \int NMP : rad., erit BV : B\Phi, id est, celeritas

mobilis in M ad celeritatem initialem, in A, sicut AB^2 in $\int BAC$ ad $GB.SB$ in $\int NMP$.

Unde, si vA significet velocitatem in A, erit BV , seu celeritas in M, quam etiam exprimam per $vM = AB^2.vA.\int BAC : GB.SB \int NMP$.

VI. Porro, quia $MX.MR = MZ : MR$, seu,

$$MX : MZ = rad : \int MXZ \text{ vel } \int NMP, \& (\S.154) MZ.MR = vM^2 ;$$

erit ex æquo $MX.MR : vM^2 = rad. : \int NMP$, and (num.V.)

$$vM^2 : vA^2 = AB^4.\int BAC^2 : GB^2 : SB^2.\int NMP^2 ;$$

& (ponendo AR & A pro radio osculi, & sollicitatione centrali initio motus in curvæ puncto A) $vA^2 : A.AR = \int BAC : rad$. Ergo ex æquo & per compositionem rationum

$$MX.MR : A.AR = AB^4.\int \overline{BAC^3} : SB^2.GB^2.\int \overline{NMP^3}.$$

Assumatur quædam magnitudo M, talis, ut M.MR sit ad A.AR, sicut

$AB^2.\int \overline{BAC^3}$ ad $GB^2.\int \overline{NMP^3}$, seu,

$$M = A.AR.AB^2.\int BAC^2 : MR.GB^2.\int NMP^3,$$

& reperietur $MX : M = AB^2 : SB^2$, id est, sollicitatio centralis in aëre resistente est in quolibet curvæ puncto M, $= M.AB^2 : SB^2$, cum scilicet mobile ex A profectum curvæ arcum AM descendendo, seu ad curvam nNB accedendo describit.

VII. Sin vero ascendat super arcum MA ex M versus A, id est a curva nNB recedendo, invenietur hoc casu $\Theta\Phi VW = \Theta\Phi\Pi\Delta - GIKH + ASTD$, atque adeo, ut paragraphi 606, & eorum quæ ia præcedentibus numeris hujus ostensa sunt, $MX = M.SB^2 : AB^2$. Quæ omnia erunt invenienda.

608. Notandum autem literam $\int$ in hac propositione esse notam characteristicam sinuum

illorum angulorum, quibus præfigitur, & $\int \overline{BAC^2}, \int \overline{NMP^3}$ significare cubos ex sinibus

angulorum BAC, NMP, & reliqua signa, scilicet A, denotare sollicitationem gravitatis centralem in curvæ puncto initiali A, & AR (quam in figura confusionis vitandæ gratia non expressi) radium circuli osculatoris, seu ejusdem curvæ cum curva in eodem puncto A. Ratio vero SB ad AB, in qua SB semper major ponitur quam AB, eadem est cum ratione abscissarum alicujus quadrilinei hyperbolici ASTD (constr.) æqualis aræ BβYQ scalæ densitatum aëris. Propterea terminini ipsi rationis SB ad AB, ad nullam determinatam magnitudinem restricti sunt, sed, pro re nata, modo majores minoresve accipi possunt, dummodo ratio inter hos terminos ea maneat, quæ est inter abscissas, cujuscunque trapezii hyperbolici prædictæ area in scala densitatum aëris æqualis.

COROLLARIUM I.

609. Adeoque in medio nullius densitatis, id est, in vacuo evanscent areæ B β YQ vel ASTD, atque adeo ratio SB ad AB mutabitur in rationem æqualitatis, fietque

$$vM : vA = AB \cdot \int BAC : GB \cdot \int NMP. \text{ Et } MX (= M \cdot AB^2 : SB^2, \text{ vel } M \cdot SB^2 : AB^2) = M.$$

Idcirco M, cujus valor jam (§. 607. num. VI.) determinatus est; exponit generalissime sollicitationem gravitatis centralem in vacuo.

COROLLARIUM II.

610. Si mobile M sollicitationibus quotcunque lateralibus, My, M2y, M3y, M4y, &c. ad diversa centra $\omega, 2\omega, 3\omega, 4\omega$, &c. simul agentibus urgetur; earum media directio MN, quæ (§. 603 & 49.) per centrum gravitatis Ω omnium punctorum y, 2y, 3y, 4y; &c. transibit, continget aliquam curvam NN, quam proinde curvam mediarum directionum sollicitationum lateralium vocare licet. Idcirco, quia (§.603.) $MX = n \cdot M\Omega$ (§.607.num. VI. & VII.)

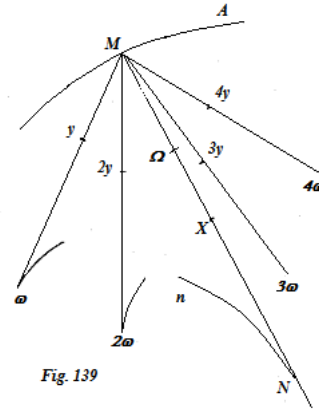


Fig. 139

$= M \cdot AB^2 : SB^2$ vel $= M \cdot SB^2 : AB^2$; hoc in casu ascensionis, illud vero in casu descensus mobilis in curva AM, erit $M\Omega : M = AB^2 : n \cdot SB^2$ in primo

casu, qui est descensus, vel $M\Omega : M = SB^2 : n \cdot AB^2$. Adeoque circumstantia problematis, sollicitationum acceleratricium vel retardatricium ad plura diversa centra simul agentium, continetur jam in problemate harum sollicitationum, si earum meta seu centrum in quolibet mobili describendæ curvæ puncto variabile est.

COROLLARIUM III.

611. Quia (§.607. num. I.) assumimus $AB \cdot AD \cdot R = D \cdot BV^2$, & (§.154.) $BV = MZ \cdot MR$, erit $R : MZ = D \cdot MR : AB \cdot AD$. Atqui $MZ : MX = NP : NM = \int NMP : \text{rad.}$ ergo ex æquo

resist.R, ad sollicit. centr. $MX = D \cdot MR \cdot \int NMP : AB \cdot AD \cdot \text{radium.}$

Item quia $MX : ZX = MN : MP = \text{rad.} : \int MNP$, erit iterum ex æquo resistentia R ad sollicitat. gravitatis tangentialem

$$ZX = D \cdot MR \cdot \int NMP : AB \cdot AD \cdot \int MNP = D \cdot MR \cdot \text{tang.} NMP : AB \cdot AD \cdot \text{radium.}$$

MX : A = MN^{2h-3} : AB^{2h-3}. Est propterea vis centripeta, id est, *solicitatio* centralis MX in logarithmica spirali ubique, ut MN^{2h-3}, si corpus descendendo arcum AM describit; & reciproce, ut MN^{2h+3} ascendendo incedit per M versus A super arcum MA.

$$\text{AR}:\overline{\text{FAC}^3}:\text{MR}:\overline{\text{NMP}^3}=\text{AF}^3:\text{MG}^3. \text{ Ergo } \text{AF}^2:\text{SF}^2=\text{AF}^3:\text{MG}^3 \text{ per consequens}$$

$$2.ASTD = 3.AOLD, \text{ vel } 2.ST.Ss = 3.OL.Oo,$$

$$\& D(= ST.Ss : Mm) = 3.OL.Oo : 2Mm,$$

vel (propter triangulorum $Mm\mu$ & MHG similitudinem)

$$= 3OL.GH : 2AF = 3OL.OF.GH : 2.OF. AF = 3AF.AD.GH : 2OF.AF$$

$$= 3AD.GH : 2OF = 3AD. \text{ tang. } AFM : 2rad.$$

Est ergo densitas in M, ut tangens anguli AFM. Vel etiam $D = 3AD. \text{ rad.} : 2.\text{tang.} MFE$,
 vel $NMP = 3.AF : AD.\text{rad.} : 2AF. \text{ tang. } NMP$.

Est vero $R : ZX$ (§. 611.) $= D.MR.\text{tang. } NMP : AB. AD.\text{rad.}$ (vel hoc loco, ubi AF
 assumenda loco AB in Fig. 138.) $= D.AF.\text{tang. } NMP : AF.AD.\text{rad.}$ (seu substituto valore
 ipsius D) $= 3AF.\text{tang. } NMP : 2AF.\text{tang. } NMP = 3 : 2$. Propterea etiam

$$R : MX = 3 \int MFA : 2.\text{rad.} = 3.GH : 2MF.$$

Quae omnia inventis Bernoullianis (Act. Lips. 1713. mens. Febr. pag. 93.) & partim
 Newtoniani conformia sunt, etsi ex alio fundamento deducta. Loquor de casu
 descensus, quandoquidem casus ascensus in circulo impossibilis est, ut eleganter ab
 Illust. Newtono animadversum; etenim foret hoc casu resistentia ad gravitatem, ut
 $-3GH$ ad $2MF$, id est, ut magnitudo privativa ad positivam.

COROLLARIUM VII.

617. Corollarium VI. nobis etiam præbet fundamentum inveniendi constructionem curvæ
 projectilium in aëre uniformis densitatis, sed resistentis in duplicata ratione celeritatum
 mobilis, posita scilicet, ut ibi, gravitate etiam uniformi & directiones ejus supponendo
 parallelas. Nam facta quadam

$$Q = f - 2dm\sqrt{(1+mm)} : m^3, \text{ erit } y = AF + \int dm : m^3Q, \& FG = x = \int dm : mmQ.$$

SCHOLIUM III.

618. Problema, quod in particulari tantum hypothesi resistentiarum aëris secundum
 compositam rationem densitatum & duplicatæ velocitatis, solutum exhibuimus,
 generalius perfici poterat pro composita ratione resistentiarum ex ratione densitatum &
 quacunque multiplicata vel submultiplicata celeritarum mobilis idque tam levi opera,
 posita præcedenti problematis analysi geometrica, ut nulla mutatio in ea facienda sit,
 præterquam in sola curva $B\beta$, quæ in hypothesi generaliori non amplius est scala
 densitatum aëris, sed curva alterius generis, quod paucis est ostendendum.

619. Esto curva βY ejus naturæ, ut existente ejus abscissa BQ æquali ubique arcui curvæ
 AM, ordinata ejus QY sit ad R resistentiam aëris in curvæ puncto M, sicut rectangulum
 datum AB.AD ad BV^2 quadratum velocitatis acquisiti in M, & hæc curva $B\beta$ generaliter
 usurpanda est loco scalæ densitatum aëris antea analysi problematis adhibita, & reliqua
 sunt absolvenda prorsus ut in §. 607, scilicet areæ $B\beta YQ$ novæ curvæ βY æquale

faciendum quadrilineum hyperbolicum ASTD ; eritque etiam nunc celeritas acquisita in M, seu æ, prorsus ut supra (§. 607. num.V.) reperiebatur, hoc solo cum discrimine quod SB nunc alius magnitudinis quam citato loco futura sit, quae adhuc quærat. Ad abbreviandum assumatur $V = AB.vA.sin. BAC : GB.sin.NMP$, eritque V velocitas, quam mobile in vacuo acquisivisset ex descensu super arcu curvae AM in ejus extremitate M, & vM vel $BV = AB.V : SB$ in casu descensus, vel $BV = SB.V : AB$ in casu ascensionis.

620. Jam, quia (constr.) $B\beta YQ = ASTD$, & $B\beta yq = AstD$, erit

$QY.Qq = ST.Ss$, sed $QY = AB.AD.R : BV^2$ (vel subrogando valorem

$AB.V : SB$, ipsius BV) $= AD.R.SB^2 : AB.V^2$, & $Qq : Mm$,

erit $AD.R.SB^2.Mm : AB.V^2 = ST.Ss$, vel omnia ducendo in SB,

$$AD.R.SB^3.Mm : AB.V^2 = SB.ST.Ss = AB.AD.Ss,$$

atque adeo

hinc elicietur $R.Mm : V^2 = AB^2.Ss : SB^3$. Adeoque omnia

$(R.Mm : V^2) = omnibus (AB^3.Ss : SB^3)$ id est $= (SB^2 - AB^2) : 2SB^2$, adeoque

designando omnia $(R.Mm : V^2)$ per $\int R.Mm : V^2$, erit

$$2 \int R.Mm : V^2 = (SB^2 - AB^2) : SB^2. \text{ Hinc } SB^2 - AB^2 = 1 : 1 - 2 \int R.Mm : V^2; \text{ pro}$$

casu descensus.

621. Pro casu ascensionis loco ipsius BV (§. 619.) substituere

oportet $SB.V : AB$ in QY, quæ pro utroque casu est $= AB.AD.R : BV^2$, fietque

$QY.Qq = AB^3.AD.R.Mm : SB^2.V^2 = ST.Ss$, vel ducendo iterum singula in SB;

$AB^3.AD.R.Mm : SB.V^2 = SB.ST.Ss = AB.BD.Ss$, hinc dividendo per AB.AD, invenietur

$AB^2.R.Mm.V^2 = SB.Ss$. Propterea omnia

$AB^2.R.Mm : V^2 = omnibus SB.Ss = \frac{1}{2}SB^2 - \frac{1}{2}AB^2$, ergo

$$2AB^2.\int R.Mm : V^2, = SB^2 - AB^2. \text{ Atque}$$

adeo $SB^2 : AB^2 = 1 + 2 \int R.Mm : V^2$. In casu ascensionis.

Hi sunt duo generales canones pro quacunque resistentiæ specie, sive hæc resistentia pendeat a velocitatibus & densitate aëris, sive non.

622. Sed, si resistentia sit in composita ratione ex densitate aëris & quacunque

multiplitata ratione celeritatis, id est, si R sit ut $D.BV^h$, ubi h est numerus quilibet pro

libitu assumptus. Vel, ad supplenda homogenea, sit $R = D.BV^h : AB^h$. Tam, quia (§.620.)

$$AB^2 : SB^2 = 1 - 2 \int R.Mm : V^2 \text{ pro casu descensus erit sumtis utrinque}$$

quantitatum elementis $-2AB^2.Ss : SB^3 = 2R.Mm : V^2$, vel (substituto

valore ipsius R & divisa æqualitate per -2) proveniet $AB^2 . Ss : SB^3 = V^{h-2} D.Mm : SB^h$,
 atque adeo $SB^{h-3} . Ss = V^{h-2} . D.Mm : AB^2$. Ergo omnia $V^{h-2} . D.Mm : AB^2$, id est,
 $\int V^{h-2} . D.Mm : AB^2$, = omnibus $SB^{h-3} . Ss$, id est, $= \frac{1}{h-2} SB^{h-2} - \frac{1}{h-2} AB^{h-2}$; atque adeo

hinc elicitor canon $SB = \sqrt[h-2]{(AB^{h-2}, + h - 2 \int V^{h-2} D.Mm : AB^2)}$. Pro

descensu corporis in curva AM. Simili modo reperietur.

$SB = AB^2 : \sqrt[h-2]{(AB^{h-2}, + 2 - h \int V^{h-2} D.Mm : AB^2)}$ pro casu ascensionis, cum mobile
 incedit ex M versus A.

613. Ut ergo, quæ in solutione hujus problematis sparsim inventa sunt, omnia conjunctim
 in conspectum producantur, num. V. §. 607, præbet velocitatem in

$M = AB.V : SB$ & $SB.V : AB$ hoc in casu ascensionis illud in descensu mobilis, ubi V
 est. (§. 619.) $AB.vA \int BAC$ applicatum ad $GB. \int NMP$, hoc est, expressio celeritatis

mobilis ad punctum curvæ M delati in vacuo. Propterea est celeritatis in aëre mobile
 acquisita in quolibet curvæ puncto M ad celeritatem ejusdem in vacuo in eodem curvæ
 puncto, ut AB ad SB; & celeritas mobili residua in M a medio resistente ad celeritatem in
 vacuo, ut SB ad AB, cum in curva ascendit. Et num. VI. §. 607. Est sollicitatio centralis in
 aëre ad sollicitationem gravitatis in vacuo mobili versante in utroque casu in quocunque
 curvæ puncto M, ut AB^2 ad SB^2 , cum mobile curvam AM descendendo describit, vel,
 sicut SB^2 ad AB^2 , si ascendendo, eundo per M versus A. Ipsæ vero V & M; scilicet
 velocitas in vacuo & sollicitatio centralis respectu curvæ puncti M, generalissime jam
 repertæ sunt, scilicet ut paulo ante $V = AB.vA. \int BAC : GB. \int NMP$, &

$M = A.AR. \int BAC^3 : MR. \int NMP^3$, in qua formula GB, concessis figurarum curvilinearum
 quadraturis, semper dantur. Ipsæ vero SB in paragraphis 620, 621 & 622 etiam
 generaliter exhibitæ sunt. Ex solutione nostra generalissima innumera alia corollaria
 potuissent deduce, quæ brevitatis studio suppressi.