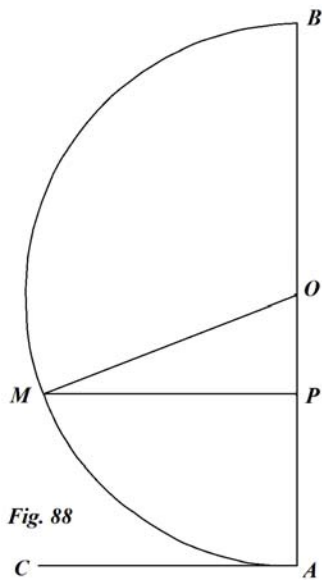
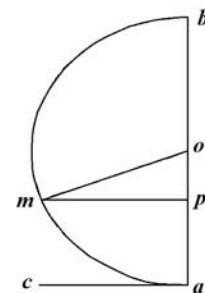


CONCERNING THE SIMILARITY AND AFFINITY OF CURVED LINES

435. In any equation for a curved line, constant quantities must be present besides the orthogonal coordinates x and y , either one or several such as a, b, c etc., by which constant lines may be designated and which with the more variable x and y constitute a number of lines of the same dimension. For if in one term a product may stand out from n lines multiplied by each other in turn, it is necessary, that in the individual terms remaining just as many lines may be multiplied by each other in turn, because otherwise heterogeneous quantities must be compared between each other, which cannot happen. On account of which in any equation for a curved line the constant lines a, b, c etc. constitute a number with the same dimensions as the variable x and y everywhere, except perhaps a certain constant line may be expressed by unity or by some other absolute number. Therefore with this notation, if no constant lines may be present in an equation, then the variables x and y alone everywhere fulfill a number of the same dimensions and thus constitute a homogeneous function. Moreover now above we have seen an equation of this kind not to relate to a curved line, but show some right lines intersecting among themselves at the same point.



436. Therefore we will consider an equation, in which besides the two variables x and y a single constant line a shall be present ; thus so that the three lines a, x and y may constitute a number of the same dimensions everywhere. An equation of this kind, provided the constant lines a can be given each and every value, may produce an infinity of curved lines, which differ from each other by magnitude only, truly in general moreover they shall become similar to each other. Therefore all curved lines, which may be understood in this manner from the same equation, deservedly are referred to the same genus and are agreed to be similar to each other nor can any one of these be taken as different, unless as it may be understood to be in circles of different magnitudes.



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437. So that this similarity may be understood better, we will consider a determined equation which besides the variables x and y , contains the single constant line a , that can be called a *parameter*, satisfying this :

$$y^3 - 2x^3 + ayy - aax + 2aay = 0.$$

Let (Fig. 88) AC be the value of the parameter a ; and with $AC = a$ being present AMB shall be the curved line satisfied by this equation, with the right line AB taken for the axis and with the coordinates called $AP = z$ and $PM = y$. Now some other value $ac = a$ may be attributed to the parameter a (Fig. 89) and amb shall be the curved line, which that equation now presents, and these curved lines AMB and amb shall be similar to each other. For if there may remain $AC = a$, $AP = x$, $PM = y$ and there shall be

$$ac = \frac{1}{n} AC = \frac{a}{n}, \text{ then truly taking } ap = \frac{1}{n} AP = \frac{x}{n}, \text{ there will be } pm = \frac{1}{n} PM = \frac{y}{n};$$

in so much as if in that equation $\frac{a}{n}$, $\frac{x}{n}$ and $\frac{y}{n}$ respectively are written in place of a , x and y , on account of all the terms divided by n^3 , that same equation will result.

438. Therefore similar curves will have this property, from which the nature of the similarity there will become clearer, so that with the abscissas AP , ap taken in the ratio of both the parameters AC and ac as well as the applied lines PM and pm likewise will be had in the same ratio, clearly if there may be taken

$$AP : ap = AC : ac,$$

then also there will be

$$PM : pm = AC : ac.$$

Therefore since there shall be

$$AP : PM = ap : pm,$$

these curves will be similar to each other in the geometrical sense, with the magnitude excepted, evidently will enjoy the same conditions. Without doubt with the homologous abscissas taken AP , ap or with the proportional parameters AC and ac , not only the applied lines PM and pm will satisfy the ratio of the parameters, but also all the other lines drawn similarly, so also the arcs of the curves AM to am will be as AC to ac . Then truly also similar areas APM and apm will be in the square ratio, or as AC^2 to ac^2 . And if any two homologous points O and o may be taken, thus so that there shall be $AO : ao = AC : ac$, and from these within the equal angles AOM , aom the right lines OM and om are drawn to the curves, and also there will be

$$OM : om = AC : ac.$$

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On account of the similitude finally also the tangents at the homologous points M and m to the axis will be inclined equally, and thus will satisfy the same radii of osculation at that place of the parameters AC and ac .

439. Hence it is apparent all circles to be similar figures, which may be satisfied by the equation $yy = 2ax - xx$; and in a like manner all the curves contained in the equation $yy = ax$, that is all parabolas, will be similar figures to each other. Moreover from equations of this kind, from which we have seen similar curves to be contained, because the coordinates x and y with the parameter a everywhere constitute the same number of dimensions, if the value of y may be defined, it will be found this equal to a homogeneous function of one dimension of a and x . Therefore in turn, if P may denote a homogeneous function of one dimension of a and x , the equation $y = P$ will be satisfied by innumerable similar curves, which arise, if each and every value successively be attributed to the parameter a . Moreover in a similar manner from an equation of this kind for curves with the same abscissas x will be equal to a function of one dimension of a and y and the parameter a itself will be equal to a function of one dimension of x and y .

440. Moreover from some given curve AMB infinitely through easy practice many others similar to amb itself can be described. For some ratio may be taken, that the homologous sides of the given curve and required to be described must satisfy between themselves, which shall be $1:n$, and, if the given curve AMB may refer to the axis AB through the normal coordinates AP and PM , upon the similar axis ab the abscissa ap may be taken, so that there shall be $AP:ap = 1:n$, and from p the applied normal pm may be erected, so that equally there shall be $PM:pm = 1:n$, and there will be the point m on the similar curve amb , thus so that the points M and m shall be homologous. Or also it will be resolved by describing from some fixed point O ; for with a curve being described from some fixed point o the angle aom always is made equal to the angle AOM and om may be cut off, so that there shall be

$$OM : om = 1 : n,$$

and the point m equally on the similar curve amb . And thus in this way, for whatever the ratio $1:n$ assumed arbitrarily, a similar curve will be able to be described. Moreover, towards this end mechanical instruments are accustomed to be constructed, with the aid of which figures of any magnitude which shall be given similar are able to be traced out.

441. But if therefore the nature of a proposed curved line AM may be expressed by some equation between the coordinates $AP = x$ and $PM = y$, thence by an easy operation the equation will be found for the similar curve am . For the homologous abscissa will be $ap = X$ and the applied line $pm = Y$; from the construction there will be $x:X = 1:n$ et $y:Y = 1:n$, from which there becomes

$$x = \frac{X}{n} \quad \text{and} \quad y = \frac{Y}{n}.$$

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Therefore these values substituted into the given equation in x and y will produce an equation between X and Y for similar curves. Therefore if in the new equation only the coordinates X and Y may be agreed to be put in place with n dimension letters, then number of dimensions everywhere will be zero ; or, if the equation is required to have fractions removed on multiplying by a certain power of n , an equation will arise, in which these three quantities X , Y and n will produce an equation with the same number of dimensions everywhere. Moreover above we have seen in every equation for similar curves both the coordinates with that constant, with the variation of which similar curves are present, and put in place a number of the same dimensions everywhere ; which therefore is the criterion of equations satisfying similar curves.

442. Just as in similar curves the abscissas and the homologous applied lines may be increased or decreased in the same ratio, thus, if the abscissas may follow another ratio, truly the applied lines another, similar curves will no longer arise. Yet truly, which curves generated in this way preserve an affinity among themselves, we will call these curves *affine* ; therefore an affinity can be regarded as a kind of similarity, certainly affine curves will become similar if both these ratios, which the abscissas and the applied lines follow separately, become equal. Therefore from some give curve AMB innumerable affine curves (Fig. 88 and 89) amb will be found in this manner : the abscissa ap is taken, thus so that there shall be $AP : ap = 1 : m$; then the applied line pm is put in place, so that there shall be $PM : pm = 1 : n$; and thus, with the changing of either one or the other of these ratios $1 : m$ and $1 : n$, innumerable curves will be produced, which will be affines of the first curve AMB .

443. The nature of a given curve AMB may be expressed by some equation between the orthogonal coordinates $AP = x$ and $PM = y$ and in the affine curve amb in the manner described the abscissa may be put $ap = X$ and the applied line $pm = Y$, on account of

$$x : X = 1 : m \text{ and } y : Y = 1 : n$$

there will be

$$x = \frac{X}{m} \text{ and } y = \frac{Y}{n}.$$

But if therefore these values may be substituted into the equation given between x and y , the general equation between X and Y for the affine curve arises. Towards setting out the nature of this equation more carefully we may put the equation for the given curve ABM thus to be performed, so that the applied line y may be equal to some function of x , which shall be $= P$, or to be $y = P$. If therefore in place of x at P , $\frac{X}{m}$ may be substituted, P becomes a function of zero dimensions of X and m thus the general equation for affine curves thus will be prepared, so that $\frac{Y}{n}$ will be equal to a function of zero dimensions of

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X or m ; or, what amounts to the same, a function of zero dimensions of Y and n will be equal to a function of zero dimensions of X and m .

444. Moreover a distinction between similar and affine curves is to be observed here chiefly, so that curves, which are similar with respect to one axis or to a fixed point, shall become the same similar curves with respect to some other axes or homologous points. But the curves, which are affine only, are such only with respect to their axis, to which they are referred, and nor for argument's sake are other axes or points themselves given, to which an affinity shall be able to refer. Truly moreover it is required to be observed, as all similar curves are referred to the same order and thus to the same kind of lines, thus also affine curves always are required to be understood of lines in the same order and of the same kind. So that which may be perceived more clearly, it will be appropriate to illustrate similitude and affinitude with several of the better known examples of curves.

445. Therefore a curve may be give related to the diameter of a circle, the nature of which may be given by the equation $yy = 2cx - xx$. There may be put

$$x = \frac{X}{n} \quad \text{and} \quad y = \frac{Y}{n},$$

and the resulting equation between X et Y will include all similar curves ; moreover there will be

$$\frac{YY}{nn} = \frac{2cX}{n} - \frac{XX}{nn}$$

or

$$YY = 2ncX - XX ;$$

from which it is apparent all curves similar to the circle to be circles also, so that the diameters of which $2nc$ differ in some manner. But for finding affine curves to the circle there may be put

$$x = \frac{X}{m} \quad \text{and} \quad y = \frac{Y}{n}$$

and there will be produced

$$\frac{YY}{nn} = \frac{2cX}{m} - \frac{XX}{mm}$$

or

$$mmYY = 2mnncX - nnXX ,$$

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which is the general equation for an ellipse related to another principal axes ; from which it is understood that all ellipses are affine curves to the circle. Whereby all ellipses are also affine curves between themselves. Also in a like manner it may be understood that all hyperbolas are affine curves between themselves. But ellipses and also hyperbolas will be similar, in which the two principal axes exist in the same ratio.

446. Because the parabola may be expressed by the equation $yy = cx$, it is evident indeed that all curves similar to it also are parabolas and thus all parabolas are curves similar to each other. Because if moreover we may consider curves affine to the parabola, on putting

$$y = \frac{Y}{n} \quad \text{and} \quad x = \frac{X}{m}$$

the equation $YY = \frac{nnc}{m} X$ will be produced, which since also it shall be an equation for

parabolas, it is evident which parabolic curves shall be affines, likewise the same parabolas are similar ; thus so that in this case both similitude and affinitude shall be generally apparent. The same too comes about in all curves, the nature of which is expressed in equations with two terms only with a constant ,

$y^3 = ccx$, $y^3 = cxx$, $yyx = c^3$ etc. are of which kind ; doubtless since these curves, both parabolas as well as hyperbolas, which are affines to other curves, the same also are similar; which agreement does not have a place in curves of other kinds, as now we have noted from the circle and the ellipse.

447. Just as from a given equation between x and y , as some number of constant quantities a, b, c etc. may be present, if values may be given to the individual constants, a single determined curves line arises, thus, if from one of the constants, for example a , it may be assumed changeable and successively from that each and every value will be given, since from some single value an individual curve arises, generally infinitely many curves will arise, which are similar, if besides a no other constant lines may be present in the equation ; otherwise truly they shall not be similar. But if besides a another changeable constant b may be put in place, then on account of the change of b from any value of a an infinitude of lines appear and thus generally from the changeability of the two constants a and b an infinity of an infinity of different curved lines will arise. If in addition an extra third changeable constant c may be assumed, then at this point infinitely more curved lines will result ; and thus, so the greater the number of constants, which may be put in place to be varied, from that the number of resulting curved lines will be expressed by a greater power of infinity.

448. But we may consider these infinite curved lines with a little more care, which emerge from one equation, while only one of the constant lines is assumed changeable . Moreover an equation of this kind, if the same axis and the same starting point of the abscissas may be retained, not only shows these curved lines, but also it indicates the

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$$y = a + \sqrt{(2cx - xx)},$$

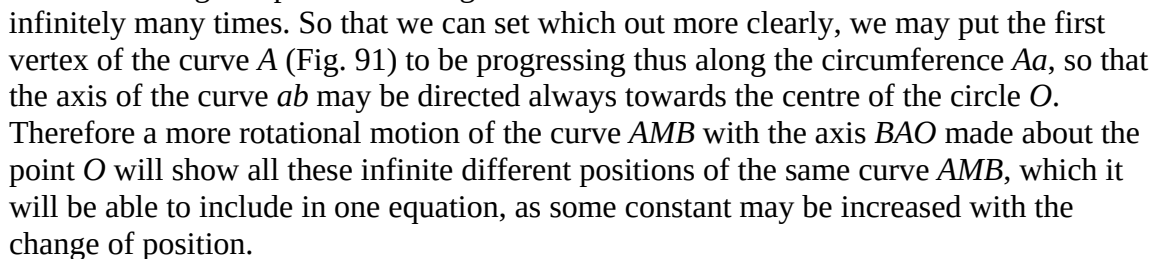
which will be the general equation of all the circles following the directrix AaL being put in place together in the manner described. Clearly all these circles will be produced from the circle found, if the line a , on which likewise A depends, may be assumed to change.

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$AK = a$ and $Ka = A$, thus so that A shall be a certain function of a , the abscissa will be called $AP = x$ and the applied line $Pm = y$, there will be $t = x - a$ and $u = y - A$. But if therefore these values may be substituted in place of t and u into the given equation between t and u , a general equation will be obtained including jointly all the curves amb . For any determined value that may be given to a , a single certain curve amb will be produced from the infinitude, which have been described by this motion. Thus, if the curve amb were a parabola expressed by the equation $uu = ct$, then infinitely many equal parabolas, of which the vertices have been set out by the directrix AaL and the axes parallel to the line AB , will be satisfied by this equation

451. Just as here we have put the vertex of the curve A thus to be advanced on a given curved directrix, so that it will remain always parallel to its axis, thus also, while the vertex may be transported along a given curve, the position of the axis of the curve ab will be able to be varied in some manner ; and thus a much more general equation will be obtained for the same curve described in a given plane following some law



452. The unchanging radius may be put to be $AO = aO = c$ and the angle shall be $AOa = \alpha$, which is assumed changeable ; from the curve described at some given situation amb , from some point m the applied line mP may be sent to the right line OAB taken for the principal axis and there shall be $OP = x$ and $Pm = y$. Then from m some perpendicular mp may be sent to the proper axis of the curve ab ; and on calling

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$$ps = x \cdot \sin. \alpha, \quad Op - Ps = x \cdot \cos. \alpha;$$
$$Pms = AOa = \alpha,$$
$$P_s = y \cdot \sin. \alpha \quad \text{and} \quad m_s = y \cdot \cos. \alpha .$$
$$Op = c + t = x \cdot \cos. \alpha + y \cdot \sin. \alpha \quad \text{and} \quad mp = u = y \cdot \cos. \alpha - x \cdot \sin. \alpha.$$
$$t = x \cdot \cos. \alpha + y \cdot \sin. \alpha - c \quad \text{and} \quad u = y \cdot \cos. \alpha - x \cdot \sin. \alpha$$

(Fig. 92), meanwhile truly with the position of the axis ab thus will be changed continually, so that the angle AOa may depend in some manner on the point a . Clearly, with the vertex situated at a , there shall be $AK = a$ and $Ka = A$, and the angle $AOa = \alpha$, where on account of the given directrix A will be some known function of a , but the sine or cosine of the angle α shall equally be some function of a . With these in place there will be

Fig. 92

From some point m of the curve amb in the first place the perpendicular mP may be sent to the principal axis AO , then truly also mp to its proper axis, and there shall be $AP = x$, $Pm = y$ and $ap = t$, $pm = u$, and an invariable equation will be given between the coordinates t and u , from which a variable equation between x and y must be able to be defined including all the curves amb .

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454. Towards this admirable end, from P the normal Ps may be drawn to mp , which will be parallel to the axis of the curve abO ; and on account of the angle $Pms = AOa = \alpha$ there will be

$$Ps = y \cdot \sin. \alpha \quad \text{and} \quad ms = y \cdot \cos. \alpha.$$

Then on account of

$$OP = a + \frac{A}{\text{tang. } \alpha} - x$$

there will be

$$ps = a \cdot \sin. \alpha + A \cdot \cos. \alpha - x \cdot \sin. \alpha$$

and

$$Op - Ps = a \cdot \cos. \alpha + \frac{a \cdot \cos. \alpha}{\text{tang. } \alpha} - x \cdot \cos. \alpha.$$

Hence there will be

$$Op = a \cdot \cos. \alpha + \frac{A \cdot \cos. \alpha}{\text{tang. } \alpha} - x \cdot \cos. \alpha + y \cdot \sin. \alpha = \frac{A}{\sin. \alpha} - t$$

and thus

$$t = A \cdot \sin. \alpha - a \cdot \cos. \alpha + x \cdot \cos. \alpha - y \cdot \sin. \alpha$$

and

$$u = -a \cdot \sin. \alpha - A \cdot \cos. \alpha + x \cdot \sin. \alpha + y \cdot \cos. \alpha.$$

On account of which, if

$$t = (x - a) \cdot \cos. \alpha - (y - A) \cdot \sin. \alpha$$

and

$$u = (x - a) \cdot \sin. \alpha + (y - A) \cdot \cos. \alpha,$$

may be substituted into the given equation between t and u , the equation sought between x and y will arise. Therefore by whatever law the same curve amb may be described in infinitely many ways, in this manner the general equation will be found containing within it all these curves at the same time.

455. Therefore in this manner an infinite number of curves of the same kind are included in the equation, as long as the differences between themselves in turn on account of the position, which may be given between t and u , should be unchanged, nor the constant a may be changed among themselves. But if one or more constants, which are present in the equation between t and u , likewise may be assumed to depend on a , then an infinitude of different curves will be obtained, either similar or dissimilar, contained equally in the

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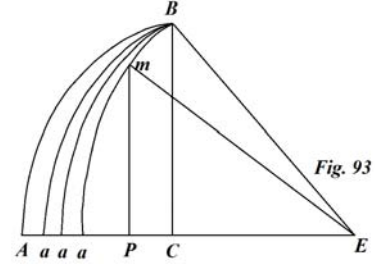
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same equation ; clearly they are all similar curves, if the equation between t and u were prepared thus, so that u is equal to some homogeneous function of one dimension of t and f , with f being some quantity depending on a ; if this does not come about, then the curves will be dissimilar.

456. So that we may illustrate the argument of the different curves by an example, we may consider (Fig. 93) an infinitude of described circles AB , aB , amB passing through the given point B , which all have their centre situated on the right line AE , the meridians of maps are accustomed to be represented by circles of this kind. From B a perpendicular may be sent to the line AC and there shall be $BC = c$, which interval is invariable.



Then some circle amB of the infinitude of circles described will be considered ; from which with one applied line mP sent, there shall be $CP = x$ and $Pm = y$, again the radius of this circle, which, even if it is constant with respect to the same circle, still with respect to all is changeable, may be put as $aE = BE = a$; there will be

$$CE = \sqrt{(aa - cc)} \quad \text{and} \quad PE = x + \sqrt{(aa - cc)}.$$

Therefore since there shall be $PE^2 + Pm^2 = aa$, the equation becomes

$$yy + xx + 2x\sqrt{(aa - cc)} + aa - cc = aa$$

or

$$yy = cc - 2x\sqrt{(aa - cc)} - xx;$$

but if the interval CE may be introduced into the equation in place of the constant variable, and there may be put $CE = a$, this equation will be had a little simpler

$$yy = cc - 2ax - xx,$$

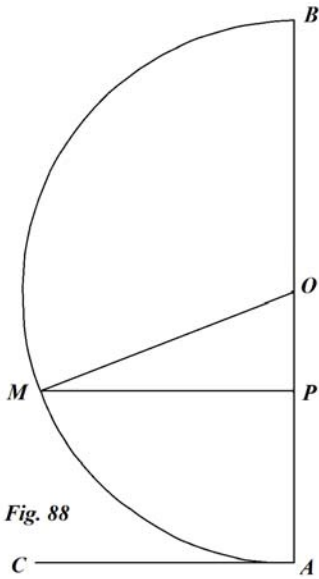
which on account of the changeability of a will show generally all the circles drawn through B and having centres on the right line AE . Truly in a similar manner any infinitude of curves established by a certain reliable law will be reduced to one equation, provided the distinction between the constant variables and the invariants will be observed properly.

CAPUT XVIII

**DE SIMILITUDINE ET AFFINITATE
LINEARUM CURVARUM**

435. In omni aequatione pro linea curva praeter coordinatas orthogonales x et y inesse debent quantitates constantes, vel una vel plures uti a, b, c etc., quibus lineae constantes designantur et quae cum variabilibus x et y ubique eundem linearum dimensionum numerum constituunt. Si enim in uno termino extet productum ex n lineis in se invicem multiplicatis, necesse est, ut in singulis reliquis terminis totidem lineae in se invicem multiplicentur, quoniam alias quantitates heterogeneae inter se comparari deberent, quod fieri non potest. Quocirca in omni aequatione pro linea curva lineae constantes a, b, c etc. cum variabilibus x et y ubique eundem dimensionum numerum constituent, nisi forte

linea quaequam constans unitate vel alio numero absoluto exprimatur. Hoc igitur notato, si nullae lineae constantes in aequatione inessent, tum variables x et y solae ubique eundem dimensionum numerum adimplerent ideoque functionem homogeneam constituerent. Supra autem iam vidimus huiusmodi aequationem ad lineam curvam non pertinere, sed aliquot rectas se invicem in eodem puncto intersecantes exhibere.



436. Contemplemur igitur aequationem, in qua praeter binas variables x et y unica insit linea constans a ; ita ut tres lineae a, x et y ubique in aequatione eundem dimensionum numerum constituent. Huiusmodi ergo aequatio, prout lineae constanti a alii atque alii valores tribuantur, infinitas producet lineas curvas, quae tantum quantitate a se invicem discreparent, ceterum vero omnino similes inter se sint futurae. Omnes ergo lineae curvae,

quae hoc modo in eadem aequatione comprehenduntur, merito ad idem genus referuntur atque inter se similes esse censentur neque aliud in illis deprehendetur discrimen, nisi quod in circulis diversae magnitudinis inesse intelligitur.

437. Quo haec similitudo melius percipiatur, consideremus aequationem determinatam praeter variables x et y unicam lineam constantem a , quam *parametrum* vocare liceat, continentem hanc

$$y^3 - 2x^3 + ayy - aax + 2aay = 0.$$

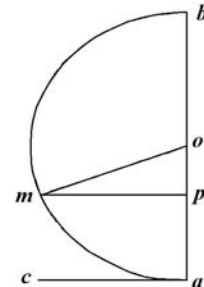


Fig. 89

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Sit (Fig. 88) AC valor parametri a ; atque existente $AC = a$ sit AMB linea curva hac aequatione contenta, sumta recta AB pro axe vocatisque coordinatis $AP = z$ et $PM = y$. Tribuatur iam parametro a quicunque alius valor $ac = a$ (Fig. 89) sitque amb linea curva, quam nunc illa aequatio praebet, eruntque hae lineae curvae AMB et amb inter se similes. Quodsi enim maneat $AC = a$, $AP = x$, $PM = y$ atque

$$\text{sit } ac = \frac{1}{n} AC = \frac{a}{n}, \text{ tum vero capiatur } ap = \frac{1}{n} AP = \frac{x}{n}, \text{ erit } pm = \frac{1}{n} PM = \frac{y}{n};$$

namque si in illa aequatione loco a , x et y scribantur respective $\frac{a}{n}$, $\frac{x}{n}$ et $\frac{y}{n}$, ob

omnes terminos per n^3 divisos eadem ipsa resultabit aequatio.

438. Curvae ergo similes hanc habebunt proprietatem, ex qua similitudinis natura eo luculentius apparebit, ut sumtis abscissis AP , ap in ratione parametrorum AC et ac applicatae PM et pm simul eandem habiturae sint rationem, scilicet si sumatur

$$AP : ap = AC : ac,$$

tum quoque erit

$$PM : pm = AC : ac.$$

Cum ergo sit

$$AP : PM = ap : pm,$$

hae curvae in sensu geometrico inter se erunt similes atque, quantitate excepta, iisdem prorsus affectionibus gaudebunt. Sumtis nimirum abscissis AP , ap homologis seu parametris AC et ac proportionalibus, non solum applicatae PM et pm rationem tenebunt parametrorum, sed etiam omnes aliae lineae similiter ductae, quin etiam curvarum arcus AM et am erunt ut AC et ac . Tum vero etiam areae similes APM et apm erunt in ratione duplicata, seu ut AC^2 ad ac^2 . Atque si sumantur duo puncta homologa O et o quaecunque, ita ut sit $AO : ao = AC : ac$, ex iisque sub aequalibus angulis AOM , aom ad curvas rectae ducantur OM et om , erit quoque

$$OM : om = AC : ac.$$

Ob similitudinem denique etiam tangentes in punctis homologis M et m ad axem aequaliter inclinabuntur atque adeo radii osculi ibidem tenebunt rationem parametrorum AC et ac .

439. Hinc patet omnes circulos esse figuras similes, quae continentur aequatione $yy = 2ax - xx$; parique modo omnes curvae aequatione $yy = ax$ contentae, hoc est omnes parabolae, erunt inter se figurae similes. Ex huiusmodi autem aequationibus, quibus curvas similes contineri vidimus, quia coordinatae x et y cum parametro a ubique eundem constituunt dimensionum numerum, si valor ipsius y definiatur, reperietur is aequalis functioni homogeneae unius dimensionis ipsarum a et x . Vicissim ergo, si denotet P

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functionem homogeneam unius dimensionis ipsarum a et x , aequatio $y = P$ innumerabiles continebit curvas similes, quae oriuntur, si parametro a successive alii atque alii valores tribuantur. Simili autem modo ex huiusmodi aequatione pro curvis similibus abscissa x aequabitur functioni unius dimensionis ipsarum a et y atque ipsa parameter a aequalis erit functioni unius dimensionis ipsarum x et y .

440. Data autem curva quacunque AMB infinitae aliae ipsi similes amb per facilem praxin describi possunt. Sumatur enim ratio quaecunque, quam latera homologa curvae datae et describendae inter se tenere debeant, quae sit $1:n$, atque, si curva data AMB referatur ad axem AB per coordinatas normales AP et PM , super axe simili ab capiatur abscissa ap , ut sit $AP:ap = 1:n$, et ex p erigatur applicata normalis pm , ut sit pariter $PM:pm = 1:n$, eritque punctum m in curva simili amb , ita ut puncta M et m sint homologa. Vel descriptio quoque ex puncto quocunque fixo O absolvi poterit; sumto enim in curva describenda puncto simili fixo o fiat perpetuo angulus aom aequalis angulo AOM et abscindatur om , ut sit

$$OM : om = 1 : n,$$

eritque punctum m pariter in curva simili amb . Hoc itaque modo, pro quavis ratione $1:n$ ad arbitrium assumpta, curva similis describi poterit. Solent autem in hunc finem confici instrumenta mechanica, quorum ope figurae cuiuscunque magnitudinis, quae sint datae similes, delineari possunt.

441. Quodsi igitur natura curvae propositae AM exprimatur aequatione quacunque inter coordinatas $AP = x$ et $PM = y$, inde facili negotio reperietur aequatio pro curva simili am . Sit enim abscissa homologa $ap = X$ et applicata $pm = Y$; erit ex constructione $x:X = 1:n$ et $y:Y = 1:n$, unde fit

$$x = \frac{X}{n} \quad \text{et} \quad y = \frac{Y}{n}.$$

Hi ergo valores in aequatione in x et y data substituti producent aequationem inter X et Y pro curvis similibus. Si igitur in hac nova aequatione solae coordinatae X et Y cum littera n dimensiones constituere censeantur, numerus dimensionum ubique erit nullus; vel, si aequatio ad fractiones tollendas multiplicetur per quampiam potestatem ipsius n , orietur aequatio, in qua tres hae quantitates X , Y et n ubique eundem dimensionum numerum producant. Supra autem vidimus in omni aequatione pro curvis similibus ambas coordinatas cum ea constante, cuius variatione curvae similes existunt, ubique eundem dimensionum numerum constituere; quod igitur est criterium aequationum curvas similes continentium.

442. Quemadmodum in curvis similibus abscissae et applicatae homologae in eadem ratione sive augentur sive diminuuntur, ita, si abscissae aliam sequantur rationem, aliam vero applicatae, curvae non amplius orientur similes. Verum tamen, quia curvae hoc

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modo ortae inter se quandam affinitatem tenent, has curvas *affines* vocabimus; complectitur ergo affinitas sub se similitudinem tanquam speciem, quippe curvae affines in similes abeunt, si ambae illae rationes, quas abscissae et applicatae seorsim sequuntur, evadant aequales. Ex curva ergo quacunque data *AMB* innumerabiles curvae affines (Fig. 88 et 89) *amb* reperientur hoc modo: sumatur abscissa *ap*, ita ut sit $AP : ap = 1 : m$; tum constituatur applicata *pm*, ut sit $PM : pm = 1 : n$; sicque, mutando harum rationum $1 : m$ et $1 : n$ vel alterutram vel utramque, innumerabiles prodibunt curvae, quae primae *AMB* erunt affines.

443. Exprimatur natura curvae datae *AMB* aequatione quacunque inter coordinatas orthogonales $AP = x$ et $PM = y$ atque in curva affini *amb* modo praecedente descripta ponatur abscissa $ap = X$ et applicata $pm = Y$, ob

$$x : X = 1 : m \text{ et } y : Y = 1 : n$$

erit

$$x = \frac{X}{m} \text{ et } y = \frac{Y}{n}.$$

Quodsi ergo hi valores in aequatione inter x et y data substituantur, proveniet aequatio generalis pro curvis affinibus inter X et Y . Ad huius aequationis naturam penitus evolvendam ponamus aequationem pro curva data *ABM* ita esse conformata, ut applicata y aequetur functioni cuicunque ipsius x , quae sit $= P$, seu esse $y = P$. Si igitur in P loco x substituatur $\frac{X}{m}$, fiet P functio nullius dimensionis ipsarum X et m ideoque

aequatio generalis pro curvis affinibus ita erit comparata, ut $\frac{Y}{n}$ aequetur functioni nullius dimensionis ipsarum X et m ; seu, quod eodem redit, functio nullius dimensionis ipsarum Y et n aequabitur functioni nullius dimensionis ipsarum X et m .

444. Discrimen autem inter curvas similes et affines hoc potissimum est notandum, quod curvae, quae sunt similes respectu unius axis vel puncti fixi, eadem similes sint futurae respectu aliorum quorumvis axium seu punctorum homologorum. Curvae autem, quae tantum sunt affines, tales tantum sunt respectu eorum axium, ad quos referuntur, neque pro lubitu alii axes seu puncta homologa in ipsis dantur, ad quae affinitas referri possit. Ceterum vero notandum est, uti omnes curvae similes ad eundem ordinem atque adeo ad idem linearum genus referuntur, ita etiam curvas affines semper in eodem linearum ordine eodemque genere comprehendendi. Quae ut clarius percipiantur, similitudinem atque affinitatem nonnullis exemplis curvarum notiorum illustrasse conveniet.

445. Sit igitur curva data circulus ad diametrum relatus, cuius natura exprimitur aequatione $yy = 2cx - xx$. Ponatur

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$$x = \frac{X}{n} \quad \text{et} \quad y = \frac{Y}{n},$$

atque aequatio inter X et Y resultans complectetur omnes curvas similes;
erit autem

$$\frac{YY}{nn} = \frac{2cX}{n} - \frac{XX}{nn}$$

seu

$$YY = 2ncX - XX ;$$

ex qua patet omnes curvas circulo similes quoque esse circulos, quorum diametri
 $2nc$ utcunque discrepent. Ad curvas autem circulo affines inveniendas ponatur

$$x = \frac{X}{m} \quad \text{et} \quad y = \frac{Y}{n}$$

prodibitque

$$\frac{YY}{nn} = \frac{2cX}{m} - \frac{XX}{mm}$$

seu

$$mmYY = 2mnncX - nnXX ,$$

quae est aequatio generalis pro ellipsi ad alterum axem principalem relata; unde
intelligitur omnes ellipses esse lineas curvas circulo affines. Quare omnes ellipses sunt
quoque curvae inter se affines. Simili autem modo intelligitur omnes hyperbolas esse
curvas inter se affines. Ellipses autem atque etiam hyperbolae, in quibus eadem ratio inter
binos axes principales intercedit, curvae erunt inter se similes.

446. Quod ad parabolam aequatione $yy = cx$ expressam attinet, perspicuum quidem est
omnes curvas ipsi similes quoque esse parabolas atque adeo omnes parabolas esse curvas
inter se similes. Quodsi autem ad curvas parabolae affines spectemus, posito

$$y = \frac{Y}{n} \quad \text{et} \quad x = \frac{X}{m}$$

prodibit aequatio $YY = \frac{nnc}{m} X$, quae cum etiam sit pro parabolis, manifestum est, quae

curvae parabolae sint affines, easdem simul parabolas esse similes; ita ut hoc casu
similitudo aequae late pateat atque affinitas. Idem quoque evenit in omnibus curvis,
quarum natura exprimitur aequatione duobus tantum terminis constante, cuiusmodi
sunt $y^3 = ccx$, $y^3 = cxx$, $yyx = c^3$ etc.; his nimirum curvis, cum parabolicis tum

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hyperbolicis, quae aliae curvae sunt affines, eadem quoque sunt similes; quae convenientia in curvis alius generis non locum habet, uti iam de circulo et ellipsi notavimus.

447. Quemadmodum ex data aequatione inter x et y , quam quotcunque quantitates constantes a , b , c etc. ingrediantur, si singulis constantibus determinati valores tribuantur, unica linea curva determinata oritur, ita, si una constantium, puta a , mutabilis assumatur eique successive alii atque alii valores tribuantur, quia ex unoquoque valore peculiaris curva nascitur, omnino infinitae curvae orientur, quae erunt similes, si praeter a nullae aliae lineae constantes aequationem ingrediantur; contra vero dissimiles. Sin autem praeter a alia quoque constans b mutabilis statuatur, tum ob mutabilitatem ipsius b ex unoquoque ipsius a valore emergent lineae curvae infinitae sicque omnino ex mutabilitate duarum constantium a et b infinities infinitae provenient lineae curvae differentes. Si insuper tertia constans c mutabilis assumatur, tum adhuc infinities plures resultabunt lineae curvae; sicque, quo maior fuerit constantium, quae mutabiles statuuntur, numerus, eo maiore infiniti potestate numerus curvarum resultantium exprimetur.

448. Consideremus autem aliquanto diligentius eas lineas curvas infinitas, quae ex una aequatione prodeunt, dum tantum una linearum constantium mutabilis assumitur. Huiusmodi autem aequatio, si idem axis idemque abscissarum initium retineatur, non solum lineas illas curvas infinitas exhibet, sed etiam earum positionem indicat, ita ut his curvis infinitis spatium quodpiam impleatur, in quo nullum assignari queat punctum, quin per id aliqua infinitarum curvarum transeat. Prout ergo aequatio fuerit comparata, curvae illae infinitae vel erunt dissimiles vel similes, uti ex praecedentibus iudicare licet; quin etiam evenire potest, ut omnes curvae sint inter se non solum similes sed etiam aequales, ratione situs tantum differentes. Sic ista aequatio

$$y = a + \sqrt{(2cx - xx)},$$

posita a mutabili, exhibebit infinitos circulos aequales radii c , quorum centra sunt in recta ad axem normali sita.

449. Hinc etiam vicissim, si una eademque curva super plano in infinitis diversis sitibus secundum certam legem describatur, aequatio praeberi poterit, qua per unius constantis mutabilitatem omnes hae infinitae curvae inter se aequales simul exhibeantur. Sit curva infinitis variis sitibus exhibita circulus (Fig. 90), cuius radius $= c$, qui ita infinities describatur, ut vertices A , a datam curvam AaL , quae *directrix* vocetur, constituent; diametri autem ab perpetuo axi AB maneant parallelae. Ad aequationem ergo pro his infinitis circulis inveniendam sumatur quodvis directricis punctum a , unde in axem

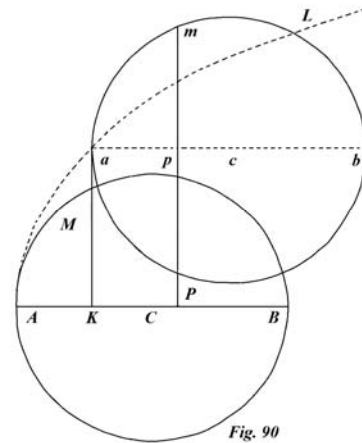


Fig. 90

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principalem demittatur perpendicularum aK . Ponatur $AK = a$; et ob directricem datam dabitur Ka per a ; sit ergo $Ka = A$, eritque A function quaeipiam ipsius a data. Tum ex a axi principali ducatur parallela ab , quae erit diameter circuli verticem in directricis puncto a habentis, ex cuius puncto quovis m ducatur applicata $mP = y$ respondens abscissae $AP = x$; erit ergo

$$ap = x - a \quad \text{et} \quad pm = y - A.$$

Positis autem $ap = t$ et $pm = u$ erit ex natura circuli $uu = 2ct - tt$; iam ob $t = x - a$ et $u = y - A$ habebitur

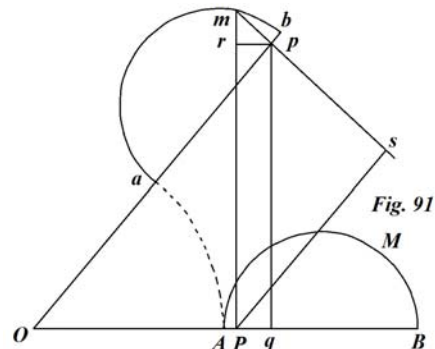
$$(y - A)^2 = 2c(x - a) - (x - a)^2,$$

quae erit aequatio generalis omnes circulos secundum directricem AaL modo descripto dispositos complectens. Omnes scilicet isti circuli ex aequatione inventa prodibunt, si linea a , a qua simul A pendet, mutabilis assumatur.

450. Simili modo, si loco circuli alia quaecunque linea curva amb ita promoveatur secundum ductum directricis AaL , ut eius vertex seu abscissarum initium a in directrice atque axis ab sibi perpetuo parallelus maneat, eadem linea curva infinities descripta habebitur atque aequatio inveniri poterit, qua omnium harum linearum curvarum natura simul comprehendatur. Data sit natura huius curvae promotae per aequationem inter coordinatas $ap = t$ et $pm = u$ ac pro axe principali, ad quem omnes curvae iunctim consideratae referantur, sumatur recta AB axibus ab parallela, quae simul sit axis directricis AaL . Posito iam, ut ante, $AK = a$ et $Ka = A$, ita ut A sit functio quaedam ipsius a , vocetur abscissa $AP = x$ et applicata $Pm = y$, erit $t = x - a$ et $u = y - A$. Quodsi ergo hi valores loco t et u in aequatione inter t et u data substituantur, obtinebitur aequatio generalis omnes curvas amb coniunctim complectens. Quicunque enim valor determinatus ipsi a tribuatur, prodibit una quaedam curva amb ex infinitis, quae per hunc motum sunt descriptae. Sic, si curva amb fuerit parabola aequatione $uu = ct$ expressa, tum infinitae parabolae aequales, quarum vertex per directricem AaL sunt dispositi axesque rectae AB paralleli, continebuntur in hac aequatione

$$(y - A)^2 = c(x - a).$$

451. Quemadmodum hic vertex curvae A in data curva directrice ita promoveri posuimus, ut eius axis sibi semper maneret parallelus, ita etiam, dum vertex per datam curvam transfertur, positio axis curvae ab utcunque variari poterit; sicque multo generalior obtinebitur aequatio pro eadem curva in dato plano secundum quamcunque legem infinities



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Ex curvae *amb* puncto quocunque *m* primum ad axem principalem *AO* demittatur perpendiculum *mP*, tum vero etiam in proprium axem *mp*, sitque $AP = x$, $Pm = y$ et $ap = t$, $pm = u$, dabiturque aequatio invariabilis inter coordinatas *t* et *u*, ex qua aequatio variabilis inter *x* et *y* omnes curvas *amb* complectens definiri debet.

454. Ad hoc praestandum ex *P* in *mp* productam ducatur normalis *Ps*, quae erit axi curvae *abO* parallela; atque ob angulum $Pms = AOa = \alpha$ erit

$$Ps = y \cdot \sin. \alpha \quad \text{et} \quad ms = y \cdot \cos. \alpha.$$

Deinde ob

$$OP = a + \frac{A}{\tan. \alpha} - x$$

erit

$$ps = a \cdot \sin. \alpha + A \cdot \cos. \alpha - x \cdot \sin. \alpha$$

et

$$Op - Ps = a \cdot \cos. \alpha + \frac{a \cdot \cos. \alpha}{\tan. \alpha} - x \cdot \cos. \alpha.$$

Hinc erit

$$Op = a \cdot \cos. \alpha + \frac{A \cdot \cos. \alpha}{\tan. \alpha} - x \cdot \cos. \alpha + y \cdot \sin. \alpha = \frac{A}{\sin. \alpha} - t$$

ideoque

$$t = A \cdot \sin. \alpha - a \cdot \cos. \alpha + x \cdot \cos. \alpha - y \cdot \sin. \alpha$$

et

$$u = -a \cdot \sin. \alpha - A \cdot \cos. \alpha + x \cdot \sin. \alpha + y \cdot \cos. \alpha$$

Quamobrem, si in aequatione inter *t* et *u* data substituantur

$$t = (x - a) \cdot \cos. \alpha - (y - A) \cdot \sin. \alpha$$

et

$$u = (x - a) \cdot \sin. \alpha + (y - A) \cdot \cos. \alpha,$$

orietur aequatio quaesita inter *x* et *y*. Quaecunque ergo lege eadem curva *amb* in plano infinities describatur, hoc modo invenietur aequatio generalis istas curvas omnes simul in se continens.

455. Hoc igitur modo in aequationem includuntur curvae numero infinitae eadem, tantum ratione situs a se invicem discrepantes, siquidem aequatio, quae inter *t* et *u* datur, fuerit invariabilis neque constantem mutabilem *a* in se contineat. Quodsi autem una

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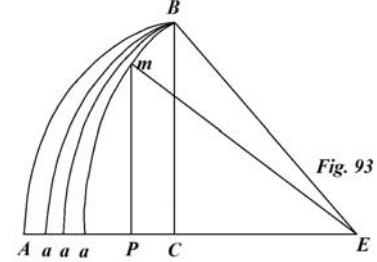
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pluresve constantes, quae in aequatione inter t et u insunt, simul ab a pendere assumantur, tum obtinebuntur infinitae curvae diversae, sive similes sive dissimiles, eadem pariter aequatione contentae ; similes scilicet erunt omnes curvae, si aequatio inter t et u ita fuerit comparata, ut u aequetur functioni cuicunque homogeneae unius dimensionis ipsarum t et f existente f quantitate utcunque ab a pendente; sin secus accadat, curvae erunt dissimiles.

456. Ut hoc argumentum curvarum diversarum exemplo illustremus, ponamus (Fig. 93) infinitos describi circulos AB , αB , amB per datum punctum B transeuntes, qui omnes centra sua habeant sita in recta AE , cuiusmodi circulis in mappis geographicis meridiani repraesentari solent. Demittatur ex B perpendicularum in rectam AC sitque $BC = c$, quod intervallum est invariabile.



Tum consideretur circulus infinitorum descriptorum quicunque amB ; unde una demissa applicata mP sit $CP = x$ et $Pm = y$, radius porro huius circuli, qui, etsi respectu eiusdem circuli est constans, tamen respectu omnium est mutabilis, ponatur $aE = BE = a$; erit

$$CE = \sqrt{(aa - cc)} \text{ et } PE = x + \sqrt{(aa - cc)}.$$

Cum igitur sit $PE^2 + Pm^2 = aa$, erit

$$yy + xx + 2x\sqrt{(aa - cc)} + aa - cc = aa$$

seu

$$yy = cc - 2x\sqrt{(aa - cc)} - xx;$$

sin autem intervallum CE loco constantis variabilis in aequationem introducatur ponaturque $CE = a$, habebitur haec aequatio aliquanto simplicior

$$yy = cc - 2ax - xx,$$

quae ob mutabilitatem ipsius a omnes omnino circulos per B ductos et centra in recta AE habentes exhibebit. Simili vero modo curvae quaecunque infinitae certa quadam lege dispositae ad unam aequationem revocabuntur, dummodo discrimen inter constantes variabiles et invariables probe observetur.